

THE SZLENK INDEX AND LOCAL ℓ_1 -INDICES

DALE ALSPACH, ROBERT JUDD, AND EDWARD ODELL

ABSTRACT. We introduce two new local ℓ_1 -indices of the same type as the Bourgain ℓ_1 -index; the ℓ_1^+ -index and the ℓ_1^+ -weakly null index. We show that the ℓ_1^+ -weakly null index of a Banach space X is the same as the Szlenk index of X , provided X does not contain ℓ_1 . The ℓ_1^+ -weakly null index has the same form as the Bourgain ℓ_1 -index: if it is countable it must take values ω^α for some $\alpha < \omega_1$. The different ℓ_1 -indices are closely related and so knowing the Szlenk index of a Banach space helps us calculate its ℓ_1 -index, via the ℓ_1^+ -weakly null index. We show that $I(C(\omega^{\omega^\alpha})) = \omega^{1+\alpha+1}$.

1. INTRODUCTION

If X is a separable Banach space, then one can study the complexity of the ℓ_1 substructure of X via Bourgain's ℓ_1 ordinal index $I(X)$, [Bo] (defined precisely below). One has $I(X) < \omega_1$ if and only if ℓ_1 does not embed into X . It was shown in [JO] that $I(X) = \omega^\alpha$ for some $\alpha < \omega_1$ provided $\ell_1 \not\hookrightarrow X$. If X has a basis, then one can also define an ℓ_1 block basis index $I_b(X)$, [JO]. In this paper we introduce and study five additional related isomorphically invariant indices: $I^+(X)$, $I_b^+(X)$, $J^+(X)$, $J_b^+(X)$ and $I_w^+(X)$. The latter we call the ℓ_1^+ -weakly null index and show it is equal to the Szlenk index of X provided that ℓ_1 does not embed into X . The ℓ_1^+ -index $I^+(X)$, and ℓ_1^+ -block basis indices are motivated by the fundamental work of James [Ja2], and of Milman and Milman [MM], on bases and reflexivity. These results yield that the ℓ_1^+ -index is countable if and only if X is reflexive, and is equal to ω if and only if X is super-reflexive. The ℓ_1^+ -block basis index measures the “shrinkingness” of a basis. The ℓ_∞^+ -index, $J^+(X)$, and the ℓ_∞^+ -block basis index are the obvious dual notions to the ℓ_1^+ -indices, and the ℓ_∞^+ -block basis index measures the “boundedly completeness” of a basis.

All the indices are defined in terms of certain trees on X . We give the necessary background on trees in Section 2 and define the indices in Section 3. In that section we also obtain a number of results concerning these indices. In Section 4 we recall the Szlenk index and discuss its relation with the ℓ_1^+ -weakly null index. Section 5 is concerned with calculating the various indices for two

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particular collections of Banach space: the $C(\alpha)$ spaces and the generalized Schreier spaces X_α for $\alpha < \omega_1$.

2. GENERAL TREES

In this section we review the basic definitions and properties of the trees we will be using. We then construct certain specific trees. These trees will be abstract sets and may be thought of as “tree skeletons.” The nodes don’t have any meaning on their own; they merely serve as a frame on which to hang our Banach space trees.

Definition 2.1. By a *tree* we shall mean a non-empty, partially ordered set (T, \leq) for which the set $\{y \in T : y \leq x\}$ is linearly ordered and finite for each $x \in T$. The elements of T are called *nodes*. The *predecessor node* of x is the maximal element x' of the set $\{y \in T : y < x\}$, so that if $y < x$, then $y \leq x'$. An *immediate successor* of $x \in T$ is any node $y > x$ such that $x \leq z \leq y$ implies that $z = x$ or $z = y$. The *initial nodes* of T are the minimal elements of T and the *terminal nodes* are the maximal elements. A *branch* of a tree is a maximal linearly ordered subset of a tree. A *subtree* of a tree T is a subset of T with the induced ordering from T . This is clearly again a tree. Further, if $T' \subset T$ is a subtree of T and $x \in T$, then we write $x < T'$ to mean $x < y$ for every $y \in T'$. We will also consider trees related to some fixed set X . A *tree on a set X* is a partially ordered subset $T \subseteq \bigcup_{n=1}^{\infty} X^n$ such that for $(x_1, \dots, x_m), (y_1, \dots, y_n) \in T$, $(x_1, \dots, x_m) \leq (y_1, \dots, y_n)$ if and only if $m \leq n$ and $x_i = y_i$ for $i = 1, \dots, m$.

We next recall the notion of the order of a tree. Let the *derived tree* of a tree T be $D(T) = \{x \in T : x < y \text{ for some } y \in T\}$. It is easy to see that this is simply T with all of its terminal nodes removed. We then associate a new tree T^α to each ordinal α inductively as follows. Let $T^0 = T$, then given T^α let $T^{\alpha+1} = D(T^\alpha)$. If α is a limit ordinal, and we have defined T^β for all $\beta < \alpha$, let $T^\alpha = \bigcap_{\beta < \alpha} T^\beta$. A tree T is *well-founded* provided there exists no subset $S \subseteq T$ with S linearly ordered and infinite. The *order* of a tree T is defined as $o(T) = \inf\{\alpha : T^\alpha = \emptyset\}$ if there exists $\alpha < \omega_1$ with $T^\alpha = \emptyset$, and $o(T) = \omega_1$ otherwise.

A tree T on a topological space X is said to be *closed* provided the set $T \cap X^n$ is closed in X^n , endowed with the product topology, for each $n \geq 1$. We have the following result (see [De]) concerning the order of a closed tree on a Polish space.

Proposition 2.2. *If T is a well-founded, closed tree on a Polish (separable, complete, metrizable) space, then $o(T) < \omega_1$.*

A map $f : T \rightarrow T'$ between trees T and T' is a *tree isomorphism* if f is one to one, onto and an order isomorphism ($\alpha < \beta$ if and only if $f(\alpha) < f(\beta)$). We will write $T \simeq T'$ if T is tree isomorphic to T' and $f : T \xrightarrow{\sim} T'$ to denote a tree isomorphism which, for brevity, we shall simply call an *isomorphism*.

Definition 2.3. [JO] For an ordinal $\alpha < \omega_1$ a tree S is a *minimal tree of order α* if for each tree T of order α there exists a subtree $T' \subset T$ of order α which is isomorphic to S . It is easy to see that if T is a tree of order β , with $\alpha \leq \beta < \omega_1$, then there exists a subtree $T' \subseteq T$ which is a minimal tree of order α . In [JO] certain minimal trees T_α for each ordinal $\alpha < \omega_1$ were constructed inductively as follows. The smallest tree T_1 is just a single node. Given T_α one chooses $z \notin T_\alpha$ and puts this as the initial element of the tree to give $T_{\alpha+1}$. Thus $T_{\alpha+1} = T_\alpha \cup \{z\}$ with $z < x$ for every $x \in T_{\alpha+1} \setminus \{z\}$. If α is a limit ordinal and T_β has been constructed for each $\beta < \alpha$, then one chooses a sequence of ordinals α_n increasing to α , and sets T_α to be the disjoint union of the trees T_{α_n} .

Definition 2.4. Let T be a tree on a set X and let T' be a subtree of T . We define another tree on X , the *restricted subtree $R(T')$ of T' with respect to T* . Let $z = (x_i)_1^n \in T'$ and let y be the unique initial node of T' such that $y \leq z$; let $m \leq n$ be such that $y = (x_i)_1^m$. If y is also an initial node of T , then set $k = 0$, otherwise let $k < m$ be such that $(x_i)_1^k$ is the predecessor node of y in T . Finally, setting $R(z) = (x_{k+1}, \dots, x_n)$, we define $R(T') = \{R(z) : z \in T'\}$. It is easy to see that $o(T') \leq o(R(T'))$.

Many of the proofs of the results we obtain rely on extracting certain subtrees from the trees we are given. To do this we construct a type of tree called a *replacement tree*. The idea is that given two trees S and S' , one can, in some sense, replace each node of S with a tree isomorphic to S' to obtain a much larger tree. We know that if a tree is isomorphic to this larger tree, for some pair S and S' , then it is easy to reverse the replacement process and obtain a subtree isomorphic to S . We discuss two specific types of replacement tree here, $T(\alpha, \beta)$ for $\alpha, \beta < \omega_1$ and $T(\alpha, s)$ for $\alpha < \omega_1$ where s is the tree which is just a countably infinite sequence of incomparable nodes.

Description 2.5. [JO] The replacement trees $T(\alpha, \beta)$ satisfy the following properties for each pair $\alpha, \beta < \omega_1$:

- (a) There exists a map $f_{\alpha, \beta} : T(\alpha, \beta) \rightarrow T_\alpha$ satisfying:

- (i) For each $x \in T_\alpha$ there exists $I = \{1\}$ or \mathbf{N} and trees $t(x, j) \simeq T_\beta$ ($j \in I$) such that $f_{\alpha, \beta}^{-1}(x) = \cup_{j \in I} t(x, j)$ (incomparable union) with $I = \{1\}$ if α is a successor ordinal and x is the unique initial node, or $\beta < \omega$, and $I = \mathbf{N}$ otherwise.
- (ii) For each pair $a, b \in T(\alpha, \beta)$, $a \leq b$ implies $f_{\alpha, \beta}(a) \leq f_{\alpha, \beta}(b)$.
- (b) $o(T(\alpha, \beta)) = \beta \cdot \alpha$.
- (c) $T(\alpha, \beta)$ is a minimal tree of order $\beta \cdot \alpha$.

The full details of the construction of these trees may be found in [JO].

Description 2.6. The trees $T(\alpha, s)$ are built up in a similar way to the minimal trees T_α except that at each stage an infinite sequence of nodes is added instead of a single node. Let $s = \{z^1, z^2, \dots\}$, an infinite sequence of incomparable nodes, and then let $T(1, s) = s$. To construct $T(\alpha + 1, s)$ from $T(\alpha, s)$ we take the set s and then after each element put a tree isomorphic to $T(\alpha, s)$. For example, $T(n, s)$ is a countably infinitely branching tree of n levels. If α is a limit ordinal, and we have constructed $T(\beta, s)$ for each $\beta < \alpha$, then we take a *sequence of successor ordinals* $\alpha_n \nearrow \alpha$ and let $T(\alpha, s)$ be the disjoint union over $n \in \mathbf{N}$ of trees isomorphic to $T(\alpha_n, s)$.

Each tree $T(\alpha, s)$ has the following properties:

1. $o(T(\alpha, s)) = \alpha$;
2. $T(\alpha, s)$ has an infinite sequence of initial nodes;
3. if z is in the derived tree $D(T(\alpha, s))$ (i.e. z is not a terminal node of $T(\alpha, s)$), then z has an infinite sequence of immediate successors.

If S is either the sequence of immediate successors of some node $z \in T(\alpha, s)$, so that $S = \{w \in T : z < w \text{ and } z \leq y \leq w \text{ implies } y = z \text{ or } y = w\}$, or the sequence of initial nodes, then we say that S is an *s-node* of $T(\alpha, s)$. In order to use the trees $T(\alpha, s)$ we must build in one more property; we need to put an ordering on the *s-nodes*. Thus, to each *s-node*, S of $T(\alpha, s)$, we associate a bijection $\psi = \psi_S : \mathbf{N} \rightarrow S$ and then we may write $S = \{z^i : i \geq 1\}$, where $z^i = \psi(i) \in S$.

Let T be a tree on a Banach space X . When we say T is *isomorphic* to $T(\alpha, s)$ we shall mean not only are they isomorphic as trees, but we shall also require that if $(x_1, \dots, x_k) \in T$, then $(x_1, \dots, x_{k-1}) \in T$. If T is such a tree and $S = \{z^i : i \geq 1\}$ is an *s-node* of T , with $z^i = (x_1, \dots, x_k, y_i)$, or $z^i = (y_i)$, for each $i \geq 1$, then we say that $\{y_i : i \geq 1\}$ is an *s-subsequence* of T .

Definition 2.7. A *weakly null tree* on a Banach space X is a tree T isomorphic to $T(\alpha, s)$, for some $\alpha < \omega_1$, such that every *s-subsequence* is weakly null.

In many of the proofs that follow we take certain subtrees of trees isomorphic to $T(\alpha, s)$ on X for some α . Given such a tree on X we assume that the sequences of nodes down a branch, and the sequences of nodes in the s -nodes satisfy some property $\mathbf{Q}(\varepsilon)$, for $\varepsilon > 0$. We take subtrees by extracting subsequences of nodes going down branches and subsequences of the s -nodes simultaneously so that these subsequences all satisfy some property $\mathbf{P}(\varepsilon)$.

The basic idea is straightforward: given a sequence $(x_i)_1^\infty$ with property $\mathbf{Q}(\varepsilon)$ we attempt to extract a subsequence $(x_{n_i})_1^\infty$ with property $\mathbf{P}(\varepsilon)$. For example, $\mathbf{Q}(\varepsilon)$ might be the property that the sequence is *normalized and weakly null* (with no dependence on ε here), while $\mathbf{P}(\varepsilon)$ could be the property that the subsequence $(x_{n_i})_1^\infty$ is an ε -perturbation of a normalized block basis of a given basis $(e_i)_1^\infty$, i.e. there exists a normalized block basis $(b_i)_1^\infty$ of $(e_i)_1^\infty$ such that $\sum_i \|b_i - x_{n_i}\| < \varepsilon$. Of course, given a normalized weakly null sequence $(x_i)_1^\infty$ in a Banach space with basis $(e_i)_1^\infty$ we can always extract a subsequence $(x_{n_i})_1^\infty$ that is an ε -perturbation of a normalized block basis of $(e_i)_1^\infty$. The trick is to do this for all sequences in a tree.

We use the same technique each time so to avoid repeating it in each proof we present the framework below for arbitrary properties $\mathbf{P}(\varepsilon)$ and $\mathbf{Q}(\varepsilon)$. Let $\varphi : \mathbf{N} \rightarrow \mathbf{N}$ be given by

$$\varphi(i + n(n-1)/2) = i, \text{ where } 1 \leq n \text{ and } 1 \leq i \leq n,$$

thus $(\varphi(n))_1^\infty = (1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \dots)$.

Lemma 2.8 (Pruning Lemma). *Let X be a Banach space, and for $\varepsilon > 0$ let $\mathbf{P}(\varepsilon)$ and $\mathbf{Q}(\varepsilon)$ be properties which a sequence (x_i) (finite or infinite) in X may possess, satisfying for every finite (or empty) sequence $(u_i)_1^l$ with property $\mathbf{P}(\varepsilon)$, and for each $\delta > 0$:*

PL(1) *for all sequences $(x_i)_1^\infty$ in X satisfying property $\mathbf{Q}(\varepsilon)$, there exists a subsequence $(x'_i)_1^\infty$ of $(x_i)_1^\infty$ such that $(u_1, \dots, u_l, x'_1, x'_2, \dots)$ has property $\mathbf{P}(\varepsilon + \delta)$ (we then say that $(x'_i)_1^\infty$ has property $\mathbf{P}(\varepsilon + \delta)$ for $(u_i)_1^l$);*

PL(2) *if $(y_{n,i})_{i=1}^\infty$ are sequences in X ($n \geq 1$) satisfying $\mathbf{Q}(\varepsilon)$ and such that $(u_1, \dots, u_l, y_{n,1}, y_{n,2}, \dots)$ has property $\mathbf{P}(\varepsilon)$ for each $n \geq 1$, then there exist sequences $(y_i)_1^\infty \subseteq \{y_{n,i} : n, i \geq 1\}$ and $1 \leq k_1 < k_2 < \dots$ with $y_i = y_{\varphi(i), k_i}$ and such that $(u_1, \dots, u_l, y_1, y_2, \dots)$ has property $\mathbf{P}(\varepsilon + \delta)$.*

PL(3) *if $(x_i)_1^\infty$ has $\mathbf{P}(\varepsilon)$, then $(x_i)_1^k$ has $\mathbf{P}(\varepsilon)$ for every $k \geq 1$.*

Then for any $\varepsilon, \delta > 0$, for every finite sequence $(u_i)_1^l$ with property $\mathbf{P}(\varepsilon)$, for every $\alpha < \omega_1$, and for every tree T on X isomorphic to $T(\alpha, s)$, if every s -subsequence of T satisfies $\mathbf{Q}(\varepsilon)$, then there exists a subtree S of T which is also isomorphic to $T(\alpha, s)$, and such that for all nodes $z = (x_i)_1^k \in S$ with immediate successors $z^j = (x_1, \dots, x_k, y_j)$, the sequence $(u_1, \dots, u_l, x_1, \dots, x_k, y_1, y_2, \dots)$ has

property $\mathbf{P}(\varepsilon + \delta)$, and the sequence $(u_1, \dots, u_l, w_1, w_2, \dots)$ has property $\mathbf{P}(\varepsilon + \delta)$, where $z^j = (w_j)$, $j \geq 1$, are the initial nodes (where the nodes z^j are ordered as an s -node of S).

Remark 2.9. We sum up the conclusion of the Pruning Lemma by saying that S has property $\mathbf{P}(\varepsilon + \delta)$ for $(u_i)_1^l$, and if S has $\mathbf{P}(\varepsilon + \delta)$ for the empty sequence, then we just say that S has $\mathbf{P}(\varepsilon + \delta)$.

Proof. We use induction on α ; the case $\alpha = 1$ follows directly from hypothesis PL(1). Suppose the result is true for α , and fix $\varepsilon, \delta > 0$, $(u_i)_1^l$ with property $\mathbf{P}(\varepsilon)$ and let T be a tree on X isomorphic to $T(\alpha + 1, s)$ such that every s -subsequence of T satisfies $\mathbf{Q}(\varepsilon)$. Let $(z^i)_1^\infty$ be the sequence of initial nodes of T with $z^i = (w_i)$ for some $w_i \in X$. Using PL(1) we may find a subsequence $(w'_i)_1^\infty$ of $(w_i)_1^\infty$ such that $(u_1, \dots, u_l, w'_1, w'_2, \dots)$ has property $\mathbf{P}(\varepsilon + \delta/2)$. Let $\bar{z}^i = (w'_i)$ and set $\bar{T} = \{x \in T : x \geq \bar{z}^i \text{ for some } i\}$. This tree is still isomorphic to $T(\alpha + 1, s)$. Now for each i let $S_i = \{x \in \bar{T} : x > \bar{z}^i\}$ so that $S_i \simeq T(\alpha, s)$ and every s -subsequence of S satisfies $\mathbf{Q}(\varepsilon)$. By PL(3) (u_1, \dots, u_l, w'_i) has $\mathbf{P}(\varepsilon + \delta/2)$ and we may apply the induction hypothesis to obtain a subtree S'_i of S_i isomorphic to $T(\alpha, s)$ and having property $\mathbf{P}(\varepsilon + \delta)$ for (u_1, \dots, u_l, w'_i) . It is easy to see that the tree $S = \cup_i (S'_i \cup \{\bar{z}^i\})$ is the required subtree of T isomorphic to $T(\alpha + 1, s)$ with property $\mathbf{P}(\varepsilon + \delta)$ for $(u_i)_1^l$.

Let α be a limit ordinal and suppose the result is true for every ordinal $\beta < \alpha$. Let $(\alpha_n)_1^\infty$ be the sequence of ordinals increasing to α so that $T(\alpha, s) = \cup_n T(\alpha_n, s)$. Let $\varepsilon > 0, \delta > 0$, let $(u_i)_1^l$ have $\mathbf{P}(\varepsilon)$ and let T be a tree isomorphic to $T(\alpha, s)$ satisfying the requirements of the lemma. Let S_n be the subtree of T isomorphic to $T(\alpha_n, s)$ and let \bar{S}_n be the subtree of S_n isomorphic to $T(\alpha_n, s)$ with property $\mathbf{P}(\varepsilon + \delta/2)$ for $(u_i)_1^l$. Let $(z^{n,i})_{i=1}^\infty$ be the sequence of initial nodes of \bar{S}_n and let $z^{n,i} = (y_{n,i})$ for $n, i \geq 1$. We have that $(y_{n,i})_{i=1}^\infty$ has property $\mathbf{P}(\varepsilon + \delta/2)$ for $(u_i)_1^l$, for each $n \geq 1$, and so by condition PL(2) we can find sequences $(y_i)_1^\infty \subseteq \{y_{n,i} : n, i \geq 1\}$ and $1 \leq k_1 < k_2 < \dots$ with $y_i = y_{\varphi(i), k_i}$ and such that $(u_1, \dots, u_l, y_1, y_2, \dots)$ has property $\mathbf{P}(\varepsilon + \delta)$. Let $S'_n = \{x \in \bar{S}_n : x \geq (y_{n, k_j}) = z^{n, k_j} \text{ for some } j \in \varphi^{-1}(n)\}$. Now S'_n has $\mathbf{P}(\varepsilon + \delta)$ for $(u_i)_1^l$ and is still isomorphic to $T(\alpha_n, s)$. We now set $S = \cup_n S'_n$, then $S \simeq T(\alpha, s)$ and has property $\mathbf{P}(\varepsilon + \delta)$ for $(u_i)_1^l$. \square

Remark 2.10. This is a purely combinatorial result; it could be restated for any set X .

- (i) Note that from the construction of S in the proof we have that if $z \in S$, then $\{x \in T : x \leq z\} \subset S$. In other words, if we remove a node y from T , then we also remove every node $x \in T$ with $x > y$.

- (ii) If $\mathbf{P}(\varepsilon)$ and $\mathbf{Q}(\varepsilon)$ are such that given $(u_i)_1^l$ with $\mathbf{P}(\varepsilon)$ and $(x_i)_1^\infty$ with $\mathbf{Q}(\varepsilon)$, we can find a subsequence $(x'_i)_1^\infty$ of $(x_i)_1^\infty$ which has property $\mathbf{P}(\varepsilon)$ for $(u_i)_1^l$, then we may modify the proof above to remove the δ . Thus, if T has $\mathbf{Q}(\varepsilon)$, then we may prune it to obtain S with $\mathbf{P}(\varepsilon)$.
- (iii) Similarly, if given $\mathbf{Q}(\varepsilon)$, we can get $\mathbf{P}(\varepsilon - \delta)$ for any $\delta > 0$, then we may modify the above proof so that given T with $\mathbf{Q}(\varepsilon)$, we can prune it to obtain S with $\mathbf{P}(\varepsilon - \delta)$ for any $\delta > 0$.

3. LOCAL INDICES

In this section we introduce the local indices on a Banach space X that we shall use throughout this paper. They have very similar definitions: one forms trees on X whose nodes satisfy some property P , and then the index is the supremum over the order of the trees. There are several different properties that we shall use to produce the different indices. We first give general results on indices defined in this way, and then we discuss the specific indices we use.

In the following X will always be a separable Banach space. Let $B_X = \{x \in X : \|x\| \leq 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$ denote the unit ball and unit sphere of X , respectively. If $(x_i)_{i \in I}$ is a sequence in X for some $I \subseteq \mathbf{N}$, then let $[x_i]_{i \in I}$ be the closed linear span of these vectors. If X also has a basis $(e_i)_1^\infty$, then we define the support of $x \in X$ with respect to $(e_i)_1^\infty$ to be $F \subseteq \mathbf{N}$ if $x = \sum_F a_i e_i$ with $a_i \neq 0$ for $i \in F$. If $z = (x_1, \dots, x_n)$ is a sequence of vectors then $\text{supp}(z) = \cup_1^n \text{supp}(x_i)$. A sequence $(x_i)_1^\infty$ is an ε perturbation of a normalized block basis of $(e_i)_1^\infty$ if there exists a normalized block basis $(b_i)_1^\infty$ of $(e_i)_1^\infty$ such that $\sum_i \|b_i - x_i\| < \varepsilon$.

Definition 3.1. Each index will be defined via a property P as follows. Let $K \geq 1$ and let $P(K)$ be a property, which depends on K , that a tree T on X may satisfy. In fact we consider $P(K)$ to be a set of sequences and we say that T is a tree with property P on X , or simply a P -tree, if T is a tree on X with property $P(K)$ for some $K \geq 1$, i.e. for every $(x_i)_1^n \in T$ we have $(x_i)_1^n \in P(K)$.

For each $K \geq 1$, set the $P(K)$ index of X to be

$$I_P(X, K) = \sup\{o(T) : T \text{ is a tree on } X \text{ with property } P(K)\}$$

and then the P index of X is given by

$$I_P(X) = \sup_{K \geq 1} I_P(X, K) .$$

The next theorem contains the general result that if the property P is sufficiently well behaved, then when the index is countable it will have the value ω^α for some $\alpha < \omega_1$.

Theorem 3.2. Let $K \geq 1$, and let P be a property for finite sequences in a Banach space satisfying:

- (i) For every $(x_i)_1^m \in P(K)$, $(x_i)_1^m$ is normalized and K -basic.

- (ii) Given $L, C \geq 1$ there exists $K' = K'(K, L, C) \geq 1$ such that if $(x_i)_1^m \in P(K)$, $(y_j)_1^n \in P(L)$ and $\max(\|x\|, \|y\|) \leq C\|x + y\|$ for every $x \in [x_i]_1^m, y \in [y_j]_1^n$, then $(x_1, \dots, x_m, y_1, \dots, y_n) \in P(K')$.
- (iii) There exists $L = L(K) \geq 1$ such that for every $(x_i)_1^m \in P(K)$ and any $1 \leq k \leq l \leq m$, $(x_i)_k^l \in P(L)$.
- (iv) There exists $K'' = K''(K) \geq 1$ such that the closure of $X^n \cap P(K)$ in the product topology on X^n is contained in $X^n \cap P(K'')$ for every $n \geq 1$.

Then either $I_P(X) = \omega_1$ and there exists $(x_i)_1^\infty \subset S_X$ and $K \geq 1$ such that $(x_i)_1^m \in P(K)$ for every $m \geq 1$, or else $I_P(X) = \omega^\alpha$ for some $\alpha < \omega_1$.

The idea behind the proof is that if we have a P -tree on X of order $\omega^\alpha \cdot r$ for some $\alpha < \omega_1$ and $r \geq 1$, then we can extend this to a P -tree of order $\omega^\alpha \cdot (r + 1)$. We do this in Lemma 3.5 by extending each terminal node of the tree of order $\omega^\alpha \cdot r$ with a tree of order ω^α . In order to do this we show how to concatenate two sequences with property P in the next lemma, and in Lemma 3.4 we show how to choose a tree of order ω^α so that we can use it to extend a finite sequence with property P . Putting all this together gives us the proof of the theorem.

Lemma 3.3. *Let $X \subseteq C[0, 1]$, let $(e_i)_1^\infty$ be a monotone basis for $C[0, 1]$ with basis projections $(P_k)_1^\infty$, let $K \geq 1$ and let $(x_i)_1^m, (y_j)_1^n \subset X$ be normalized K -basic sequences. If $k \geq 1$ and $\delta, \varepsilon > 0$ are such that $2\varepsilon < \delta$, $\|(I - P_k)x\| \leq \varepsilon\|x\|$ for each $x \in [x_i]_1^m$ and $\|(I - P_k)y\| \geq \delta\|y\|$ for each $y \in [y_j]_1^n$, then for every $x \in [x_i]_1^m$ and $y \in [y_j]_1^n$*

$$\max(\|x\|, \|y\|) \leq \frac{4}{\delta - 2\varepsilon} \|x + y\|. \quad (1)$$

Proof. Let $z = \sum_1^m a_i x_i + \sum_1^n b_j y_j$, and consider the following two possibilities:

- (i) $\|\sum_1^n b_j y_j\| \geq \frac{1}{2} \|\sum_1^m a_i x_i\|$;
- (ii) $\|\sum_1^n b_j y_j\| < \frac{1}{2} \|\sum_1^m a_i x_i\|$.

In case (i),

$$\begin{aligned}
\delta \left\| \sum_1^n b_j y_j \right\| &\leq \left\| (I - P_k) \left(\sum_i^n b_j y_j \right) \right\| \\
&\leq \left\| (I - P_k) z \right\| + \left\| (I - P_k) \left(\sum_1^m a_i x_i \right) \right\| \\
&\leq 2\|z\| + \varepsilon \left\| \sum_1^m a_i x_i \right\| \\
&\leq 2\|z\| + 2\varepsilon \left\| \sum_1^n b_j y_j \right\|,
\end{aligned}$$

so that

$$\|z\| \geq \frac{\delta - 2\varepsilon}{2} \left\| \sum_1^n b_j y_j \right\| \geq \frac{\delta - 2\varepsilon}{4} \left\| \sum_1^m a_i x_i \right\|. \quad (2)$$

For case (ii)

$$\|z\| \geq \left\| \sum_1^m a_i x_i \right\| - \left\| \sum_1^n b_j y_j \right\| \geq \frac{1}{2} \left\| \sum_1^m a_i x_i \right\| \geq \left\| \sum_1^n b_j y_j \right\| \quad (3)$$

and inequality (1) now follows from inequalities (2) and (3). \square

Lemma 3.4. *Let $X \subseteq C[0, 1]$ and let $(e_i)_1^\infty$ be a monotone basis for $C[0, 1]$, with basis projections $(P_k)_1^\infty$. Let $K \geq 1$ and let T be a tree on X of order ω^α for some $\alpha < \omega_1$ such that each node $(x_i)_1^m \in T$ is a normalized K -basic sequence. Then for any $k \geq 1$ and $0 < \delta < 1/(2K)$ there exists a tree T' of order ω^α such that for each node $(y_i)_1^n \in T'$, $\|(I - P_k)y\| \geq \delta\|y\|$ whenever $y \in [y_i]_1^n$, and for each terminal node $(w_i)_1^n$ of T' there exist $l, m \geq 1$ and $(x_i)_1^m \in T$ such that $1 \leq l \leq l+n-1 \leq m$ and $w_i = x_{l+i-1}$ for $i = 1, \dots, n$.*

Proof. We may assume that $\omega^\alpha = \lim_n \omega^{\alpha_n} \cdot n$ for some sequence $\alpha_n \nearrow \alpha$. We may also assume that $T = \cup_n T(n)$ with $T(n)$ isomorphic to the replacement tree $T(n, \omega^{\alpha_n})$. It is sufficient to find a sequence $n_r \nearrow \omega$ and trees $T'_r \subseteq T(n_r)$ of order $\omega^{\alpha_{n_r}} \cdot r$ such that T'_r satisfies the conditions of the lemma for each $r \geq 1$.

Let $0 < \xi < 1/(2K) - \delta$ and let $N \geq 1$ and sets $(A_l)_1^N$ satisfy for $1 \leq l \leq N$:

$$A_l \subseteq B_{[e_i]_1^k}, \quad \text{diam } A_l < \xi, \quad \cup_1^N A_l = B_{[e_i]_1^k}.$$

Let $r \geq 1$ and choose $n \geq (N+1)r$. Consider a subtree S_1 of $T(n)$ isomorphic to $T(N+1, \omega^{\alpha_n} \cdot r)$ and let $F_1 : R(S_1) \rightarrow T_{N+1} = \{a_1, \dots, a_{N+1}\}$, with $a_1 < \dots < a_{N+1}$, be the defining map for the

replacement tree. Then $F_1^{-1}(a_1)$ is isomorphic to $T_{\omega^{\alpha_n} \cdot r}$. If $\|(I - P_k)y\| \geq \delta\|y\|$ for all $y \in [y_i]_1^j$, and every $(y_i)_1^j \in F_1^{-1}(a_1)$, then $F_1^{-1}(a_1)$ is the subtree we seek. If not, then there exists a terminal node $(y_i)_1^{k_1}$ of $F_1^{-1}(a_1)$ and $y_1 \in [y_i]_1^{k_1}$ with $\|y_1\| = 1$ and $\|(I - P_k)y_1\| < \delta$.

Let $S_2 = \{(u_1, \dots, u_j) \in S_1 : j > k_1 \text{ and } u_i = y_i^1 \ (i = 1, \dots, k_1)\}$, so that the restricted tree $R(S_2)$ is isomorphic to $T(N, \omega^{\alpha_n} \cdot r)$. Let $F_2 : R(S_2) \rightarrow T_N = \{a_2, \dots, a_{N+1}\}$ be the restriction of F_1 to $R(S_2)$ so that $F_2^{-1}(a_2) \simeq T_{\omega^{\alpha_n} \cdot r}$. Again, either $F_2^{-1}(a_2)$ is the required tree, or else there is a terminal node $(y_i)_1^{k_2}$ in $F_2^{-1}(a_2)$ and $y_2 \in [y_i]_1^{k_2}$ with $\|y_2\| = 1$ and $\|(I - P_k)y_2\| < \delta$. Continuing in this way we obtain either a subtree T'_r isomorphic to $T_{\omega^{\alpha_n} \cdot r}$ such that $\|(I - P_k)y\| \geq \delta\|y\|$ for each $y \in [y_i]_1^j$, and every $(y_i)_1^j \in T'_r$, or else there is a branch $(y_1^1, \dots, y_{k_1}^1, \dots, y_1^{N+1}, \dots, y_{k_{N+1}}^{N+1})$ of T and normalized vectors $y_j \in [y_i]_{i=1}^{k_j}$ such that $\|(I - P_k)y_j\| < \delta$ for $j = 1, \dots, N+1$. But then there exist $l, j, j' \in \{1, \dots, N+1\}$ ($j \neq j'$) such that $P_k y_j, P_k y_{j'} \in A_l$, and hence

$$\frac{1}{K} \leq \|y_j - y_{j'}\| \leq \|P_k(y_j - y_{j'})\| + \|(I - P_k)(y_j - y_{j'})\| < \xi + 2\delta < \frac{1}{K},$$

a contradiction. The last condition on T is clear from the proof. \square

Lemma 3.5. *Let $X \subseteq C[0, 1]$ and let P be a property satisfying conditions (i)–(iv) of Theorem 3.2. For all $K \geq 1$ there exists $L \geq 1$ such that for every $\alpha < \omega_1$ and $r \geq 1$, if there exists a $P(K)$ tree on X of order $\omega^\alpha \cdot r$, then there exists a $P(L)$ tree on X of order $\omega^\alpha \cdot (r + 1)$.*

Proof. Let S be the given $P(K)$ tree on X of order $\omega^\alpha \cdot r$ and let T be a $P(K)$ tree on X of order ω^α . Let $(z^j)_1^\infty$ be the sequence of terminal nodes of S , so that $z^j = (x_i^j)_{i=1}^{m_j}$. Choose $0 < \delta < 1/(2K)$, $0 < \varepsilon < \delta/2$ and for each $j \geq 1$ find $k_j \geq 1$ such that $\|(I - P_{k_j})x\| \leq \varepsilon\|x\|$ whenever $x \in [x_i^j]_1^{m_j}$. Apply the previous lemma to T for K and δ with $k = k_j$ to obtain a tree $T(z^j)$ of order ω^α such that for each node $(y_i)_1^n \in T(z^j)$ we have $\|(I - P_{k_j})y\| \geq \delta\|y\|$ whenever $y \in [y_i]_1^n$. From condition (iii) of Theorem 3.2 and the construction of $T(z^j)$ there exists $K' = K'(K)$ such that $T(z^j)$ is a $P(K')$ tree and hence by Lemma 3.3 and condition (ii) for property P , there exists $L = L(K, \delta, \varepsilon)$ such that the tree

$$S(z^j) = \{(x_1^j, \dots, x_{m_j}^j, y_1, \dots, y_n) : (y_i)_1^n \in T(z^j)\} \cup \{(x_1^j), \dots, (x_1^j), \dots, x_{m_j}^j\}$$

is a $P(L)$ tree on X . The tree $\cup_{j=1}^\infty S(z^j)$ is the required $P(L)$ tree on X of order $\omega^\alpha \cdot (k + 1)$. \square

Proof of Theorem 3.2. If $I_P(X) = \omega_1$, then, since the closure of a $P(K)$ tree is a $P(K')$ tree for some $K' \geq 1$ by condition (iv), it follows from Proposition 2.2 that there exists an infinite sequence as in the statement of the theorem.

Otherwise we assume the index is countable and let T be a $P(K)$ -tree on X of order ω^γ . By the previous lemma there exist numbers $K_i \geq 1$ and $P(K_i)$ -trees T_i on X of order $\omega^\gamma \cdot i$ for $i = 1, 2, \dots$ ($K_1 = K$). Therefore the P -index is at least $\omega^{\gamma+1}$. It follows that the P -index is

$$\begin{aligned} I_P(X) &= \sup\{\omega^\gamma \cdot k : \text{there exists } K \text{ and a } P(K)\text{-tree on } X \text{ of order } \omega^\gamma, k \in \mathbf{N}\} \\ &= \sup\{\omega^{\gamma+1} : \text{there exists } K \text{ and a } P(K)\text{-tree on } X \text{ of order } \omega^\gamma\} \\ &= \omega^\alpha \end{aligned}$$

for some $\alpha < \omega_1$. □

Definition 3.6. We shall use the following indices; we give the name of the index, the symbol we use for it and then the property P that each node of the tree must satisfy for $K \geq 1$.

ℓ_1 -index, $I(X)$:

$(x_i)_1^m$ is normalized and K -equivalent to the unit vector basis of ℓ_1^m , $(x_i)_1^m \overset{K}{\sim} \text{uvb } \ell_1^m$, i.e.

$$\frac{1}{K} \sum_1^m |a_i| \leq \left\| \sum_1^m a_i x_i \right\| \leq \sum_1^m |a_i|$$

for every $(a_i)_1^m \subset \mathbf{R}$. (Of course, this second inequality is always true.)

ℓ_1^+ -index, $I^+(X)$:

$(x_i)_1^m$ is an ℓ_1^+ - K -sequence, i.e. $(x_i)_1^m$ is normalized, K -basic and satisfies

$$\frac{1}{K} \sum_1^m a_i \leq \left\| \sum_1^m a_i x_i \right\|$$

whenever $(a_i)_1^m \subset \mathbf{R}^+$.

ℓ_∞ -index, $J(X)$:

$(x_i)_1^m$ is normalized and K -equivalent to the unit vector basis of ℓ_∞^m , $(x_i)_1^m \overset{K}{\sim} \text{uvb } \ell_\infty^m$, i.e.

there exist $c, C \geq 1$ such that $cC \leq K$ and

$$\frac{1}{c} \max_{1 \leq i \leq m} |a_i| \leq \left\| \sum_1^m a_i x_i \right\| \leq C \max_{1 \leq i \leq m} |a_i|$$

for every $(a_i)_1^m \subset \mathbf{R}$.

ℓ_∞^+ -index, $J^+(X)$:

$(x_i)_1^m$ is an ℓ_∞^+ - K -sequence, i.e. $(x_i)_1^m$ is normalized, K -basic and there exists a sequence

$(a_i)_1^m \subset [1/K, 1]$ such that $\left\| \sum_1^m a_i x_i \right\| \leq K$.

Remark 3.7. It is an easy consequence of the geometric Hahn-Banach theorem that the following is an equivalent definition of an ℓ_1^+ sequence. Let $K \geq 1$, then a sequence $(x_i) \subset X$ (finite or infinite) is an ℓ_1^+ - K -sequence if and only if (x_i) is a normalized, K -basic sequence and there exists $f \in S_{X^*}$ such that $f(x_i) \geq \frac{1}{K}$ for each i .

Rosenthal [Ro2] studied other aspects of ℓ_1^+ and ℓ_∞^+ sequences under the names wide-(s) and wide-(c) sequences respectively, with different quantifications (see Lemma 3.17). For $\lambda \geq 1$, $(x_i)_1^m$ is a λ -wide-(s) sequence in a Banach space X if it is 2λ -basic with $\|x_i\| \leq \lambda$ for $i \leq m$ and

$$\sup_{k \leq m} \left| \sum_k^m a_i \right| \leq \lambda \left\| \sum_1^m a_i x_i \right\|$$

for every $(a_i)_1^m \subset \mathbf{R}$. A λ -wide-(c) sequence in X is a λ -basic sequence $(x_i)_1^m \subset X$ with $1/\lambda \leq \|x_i\| \leq 1$ for $i \leq m$ and $\|\sum_1^m x_i\| \leq \lambda$. We investigate the relationships between these notions in Lemma 3.17 below.

Each of the above indices has a companion *block basis index*, where the property has the additional requirement that the space X have a basis $(e_i)_1^\infty$ and that each node $(x_i)_1^m$ of the tree be a block basis of $(e_i)_1^\infty$. The block basis indices are written $I_b(X)$, $I_b^+(X)$ etc. They are calculated with respect to a fixed basis $(e_i)_1^\infty$ of X and should more properly be written $I_b(X, (e_i)_1^\infty)$ and $I_b^+(X, (e_i)_1^\infty)$ etc., since they depend on the basis. For example James' space, J , has a shrinking basis $(e_i)_1^\infty$ and $I_b(J, (e_i)_1^\infty) < \omega_1$, but it also has a non-shrinking basis $(f_i)_1^\infty$, and for this we have $I_b(J, (f_i)_1^\infty) = \omega_1$. For the ℓ_1 -index, from [JO] we know that if $I(X) = \omega^{1+\alpha}$ for some $\alpha < \omega_1$, then $I_b(X) = \omega^\alpha$ or $\omega^{1+\alpha}$. It is easy to construct spaces X (see [JO] Remark 5.15 (ii)) with two different bases $(e_i)_1^\infty$ and $(f_i)_1^\infty$, such that $I(X) = \omega^{n+1}$, $I_b^+(X, (e_i)_1^\infty) = \omega^n$ and $I_b(X, (f_i)_1^\infty) = \omega^{n+1}$. However, because we shall be working with a fixed basis for X we shall omit reference to the basis. Furthermore, if the basis for X is unconditional, then $I_b^+(X) = I_b(X)$.

The trees used to calculate each index are named after the index. Thus a tree with property $P(K)$ for the ℓ_1 -index is called an ℓ_1 - K -tree, or just an ℓ_1 -tree.

Corollary 3.8 (Corollary to Theorem 3.2). *Let P be a property satisfying the conditions of Theorem 3.2. Let $(e_i)_1^\infty$ be a basis for a Banach space X , and let $I_P(X, (e_i)_1^\infty)$ be a block basis index defined via P . Either $I_P(X, (e_i)_1^\infty) = \omega_1$, and there exist $K \geq 1$ and a normalized block basis $(x_i)_1^\infty$ of $(e_i)_1^\infty$ such that $(x_i)_1^m \in P(K)$ for every $m \geq 1$, or else $I_P(X, (e_i)_1^\infty) = \omega^\alpha$ for some $\alpha < \omega_1$.*

Proof. We only need make a couple of modifications to the proof of Theorem 3.2. Instead of embedding X into $C[0, 1]$ and using a basis there we use the basis $(e_i)_1^\infty$ of X . Then in the proof of Lemma 3.5 we must ensure that we construct a block basis tree. But this is easy. In Lemma 3.5,

for each terminal node $z^j = (x_i^j)_{i=1}^{m_j}$ we choose k_j so that $P_{k_j} x_i^j = x_i^j$ ($i = 1, \dots, m_j$), and once we find the subtree $T(z^j)$ of T , we take a further subtree

$$T(z^j)' = \{(x_i)_{k_j+1}^m : k_j < m, (x_i)_1^m \in T(z^j)\},$$

which also has order ω^α . Since we started with a block basis tree T , the subtrees $T(z^j)$ and $T(z^j)'$ will also be block basis trees, as will the final tree. \square

We next note for future reference that each of the four properties we have defined are easily seen to satisfy the conditions of Theorem 3.2. The proofs are elementary calculations.

Lemma 3.9. *Each of the four properties in Definition 3.6 satisfies conditions (i)–(iv) in Theorem 3.2 for a property P .*

We have the following Corollary of Theorem 3.2, which includes some results already proved in [JO]:

Corollary 3.10. *For a separable Banach space X , each of the indices $I(X)$, $I_b(X)$, $I^+(X)$, $I_b^+(X)$, $J(X)$, $J_b(X)$, $J^+(X)$, $J_b^+(X)$ is either uncountable or ω^α for some $\alpha < \omega_1$.*

Definition 3.11. We need one more index which doesn't fit into this pattern since it relies on the structure of the trees used. For $K \geq 1$ a tree T on X is an ℓ_1^+ - K -weakly null tree if each node $(x_i)_1^m \in T$ is an ℓ_1^+ - K -sequence and T is a weakly null tree (see Definition 2.7). The ℓ_1^+ -weakly null index is written $I_w^+(X)$. We show in Theorem 3.22 that when $I_w^+(X)$ is countable it is also equal to ω^α for some $\alpha < \omega_1$.

In the light of the block basis indices above we make the following definition.

Definition 3.12. Let X be a Banach space with a basis $(e_i)_1^\infty$. A *block basis tree* on X is a tree on the unit sphere S_X of X such that every node $(x_i)_1^n \in T$ is a block basis of $(e_i)_1^\infty$.

As well as taking subtrees we also want to take “block trees”. They are an extension of the notion of a block basis to trees.

Definition 3.13. Let T be a tree on the unit sphere $S(X)$ of a Banach space X . We say S is a *block tree* of T , written $S \preceq T$, if S is a tree on $S(X)$ such that there exists a subtree $T' \subset T$ and an isomorphism $f : T' \xrightarrow{\sim} S$ satisfying: $f((x_i)_1^m) = (y_i)_1^n$ is a normalized block basis of $(x_i)_1^m$ for each $(x_i)_1^m \in T'$, and if $(x_i)_1^k \in T'$ for some $k < m$, then there exists $l < n$ such that $(y_i)_1^l = f((x_i)_1^k)$ and $(y_i)_{l+1}^n$ is a normalized block basis of $(x_i)_{k+1}^m$.

In the next theorem we give some basic properties of these indices. Statement (i) is due to Bourgain [Bo], and (ii), (iii) were proven in [JO].

Theorem 3.14. *Let X be a separable Banach space, then*

- (i) $I(X) < \omega_1$ if and only if ℓ_1 does not embed into X ;
- (ii) If $I(X) \geq \omega^\omega$, then $I(X) = I_b(X)$;
- (iii) If $I(X) = \omega^n$ for some $n \in \mathbf{N}$, then $I_b(X) = \omega^m$ where $m \in \{n, n-1\}$;
- (iv) $I^+(X) < \omega_1$ if and only if X is reflexive;
- (v) $I^+(X) = \omega$ if and only if X is super-reflexive;
- (vi) $I^+(X) = J^+(X)$ and hence $J^+(X) < \omega_1$ if and only if X is reflexive;
- (vii) $I_b^+(X, (e_i)_1^\infty) < \omega_1$ if and only if $(e_i)_1^\infty$ is a shrinking basis for X ;
- (viii) $J_b^+(X, (e_i)_1^\infty) < \omega_1$ if and only if $(e_i)_1^\infty$ is a boundedly complete basis for X ;
- (ix) If $\ell_1 \not\hookrightarrow X$, then $I_w^+(X) < \omega_1$ if and only if X^* is separable.

X is assumed to have a basis $(e_i)_1^\infty$ in (ii), (iii), (vii) and (viii).

Proof. Statements (iv) and (v) are results of James [Ja2] and Milman and Milman [MM] stated in terms of the ℓ_1^+ -index. In particular (iv) follows from [Ja2] Theorem 1 or [MM] Corollary of Theorem 2, while (v) follows from [Ja3] and [Ja4].

Before we can give the proof of (vi) we need some results on the relationship between ℓ_1^+ and ℓ_∞^+ sequences, so we shall postpone the proof until we have these..

For part (vii), if $(e_i)_1^\infty$ is not shrinking, then it has a normalized block basis which is an ℓ_1^+ sequence, giving an ℓ_1^+ -block basis tree of order ω_1 , and the other direction follows from Proposition 2.2, since X is separable and the closure of an ℓ_1^+ -block basis tree is again an ℓ_1^+ -block basis tree.

If $J_b^+(X, (e_i)_1^\infty) = \omega_1$, then there exist $K \geq 1$ and a normalized block basis $(x_i)_1^\infty$ of $(e_i)_1^\infty$ so that $(x_i)_1^m$ is an ℓ_∞^+ - K sequence for each $m \geq 1$. By a compactness argument it is easy to find $(a_i)_1^\infty \subset [1/K, 1]$ such that $\|\sum_1^m a_i x_i\| \leq 2CK$, where $C \geq 1$ is the basis constant of $(e_i)_1^\infty$. Thus $(e_i)_1^\infty$ is not boundedly complete. The converse is clear and part (viii) follows.

The proof of part (ix) requires more work. First, if X^* is separable, then by Zippin, [Z] X embeds into a Banach space with a shrinking basis. Thus it is sufficient to prove that if X has a shrinking basis $(e_i)_1^\infty$, then $I_w^+(X) < \omega_1$. If we show that $I_w^+(X) \leq I_b^+(X)$, then this would follow from part (vii). To show this we will take an ℓ_1^+ -weakly null tree on X of order α , and apply the Pruning Lemma to obtain a perturbation of an ℓ_1^+ -block basis tree on X of order α . Let T be an ℓ_1^+ -weakly null tree on X of order α , so that T is isomorphic to $T(\alpha, s)$. Define $(x_i)_1^\infty$ to have

property $\mathbf{Q}(\varepsilon)$ if it is weakly null, and define $(x_i)_1^\infty$ to have property $\mathbf{P}(\varepsilon)$ if it is an ε -perturbation of a block basis of $(e_i)_1^\infty$. Using that $(e_i)_1^\infty$ is shrinking it is standard work to show that $\mathbf{Q}(\varepsilon)$ and $\mathbf{P}(\varepsilon)$ satisfy the requirements of the Pruning Lemma, and hence we may obtain a subtree T' of order α which is a perturbation of an ℓ_1^+ -block basis tree of order α on X , i.e. each terminal node is an ε -perturbation of an ℓ_1^+ block basis of $(e_i)_1^\infty$.

Next suppose that X^* is not separable; we shall show that $I_w^+(X) = \omega_1$. Let Δ denote the Cantor set, and let $(A_{n,i})$ be a sequence of subsets of Δ for $n = 0, 1, 2, \dots$ and $i = 0, 1, 2, \dots, 2^n - 1$ such that $A_{0,0} = \Delta$ and each $A_{n,i}$ is the union of the disjoint, non-empty, clopen sets $A_{n+1,2i}$ and $A_{n+1,2i+1}$, with $\lim_{n \rightarrow \infty} \sup_{0 \leq i < 2^n} \text{diam}(A_{n,i}) = 0$. A *Haar system on Δ* (relative to $(A_{n,i})$) is a sequence of continuous functions, $(h_m)_1^\infty \subseteq C(\Delta)$ with

$$h_{2^n+i} = \mathbf{1}_{A_{n+1,2i}} - \mathbf{1}_{A_{n+1,2i+1}} \quad (n = 0, 1, 2, \dots, \quad i = 0, 1, 2, \dots, 2^n - 1) ,$$

where $\mathbf{1}_A$ is the characteristic function of the set A . A sequence of continuous functions, $(g_m)_1^\infty \subseteq C(\Delta)$ is a *Haar system on Δ up to ζ* if there exists a Haar system $(h_m)_1^\infty$ on Δ such that for each $m \geq 1$, $\text{supp } g_m = \text{supp } h_m$, $\text{sign } g_m = \text{sign } h_m$ and $1 - \zeta \leq |g_m(x)| \leq 1$ for every $x \in \text{supp } g_m$.

Given a Banach space X , and a subset $\Delta \subseteq B_{X^*}$ which is weak* homeomorphic to the Cantor set, a sequence $(x_m)_1^\infty \subseteq S_X$ is a *Haar system up to ζ , relative to Δ* if the restrictions $(x_m|_\Delta)_1^\infty$ form a Haar system up to ζ on $C(\Delta)$.

By a result of Stegall [St], since X^* is not separable, we have that given $\zeta > 0$ there exists a set $\Delta \subseteq B_{X^*}$ which is weak* homeomorphic to the Cantor set, and a dyadic tree S of elements in S_X which form a Haar system up to ζ , relative to Δ . The dyadic tree is the natural one, with $x < y$ if and only if $\text{supp } x|_\Delta \supset \text{supp } y|_\Delta$. Let $\tau_\omega = \{(n_1, \dots, n_k) : 1 \leq k, \quad n_i \in \mathbf{N}, \quad i \leq k\}$ be the countably branching tree with ω levels, ordered by $(m_1, \dots, m_k) \leq (n_1, \dots, n_l)$ if and only if $k \leq l$ and $m_i = n_i$ for $i \leq k$. Choose $S_1 \subseteq S$ isomorphic to τ_ω so that if $x \in S_1$ has immediate successors $(x_i)_1^\infty$ (i.e. if x is equivalent to the node (n_1, \dots, n_k) of τ_ω , then x_i corresponds to (n_1, \dots, n_k, i) for $1 \leq i$), then

$$\text{supp } x_i|_\Delta \subseteq \{x^* \in \Delta : x(x^*) \geq 1 - \zeta\} .$$

Since X does not contain ℓ_1 , it follows by [Ro1] that we may prune S_1 to obtain a further subtree S_2 isomorphic to τ_ω so that if $(x_i)_1^\infty$ is the sequence of immediate successors of $x \in S_2$ as above (or the sequence of initial nodes), then $(x_i)_1^\infty$ is weak-Cauchy. Furthermore they will still satisfy the property above and we may assume that if we set

$$y_j = (x^{2^{j-1}} - x^{2^j}) / \|x^{2^{j-1}} - x^{2^j}\| \quad \text{for } j \geq 1 ,$$

then $(y_i)_1^\infty$ is 2-basic and weakly null. From this we can create a tree $S_3 \subseteq S_X$ which is isomorphic to τ_ω with nodes y_i as above and letting the immediate successors of such a node y_i be formed in the same manner from the successors in S_2 of x_{2i-1} .

The resulting tree has the property that if $(z_i)_1^\infty$ is a branch of S_3 , then $\text{supp } z_{i+1}|_\Delta \subset \{x^* \in \Delta : z_i(x^*) > (1 - \zeta)/2\}$ for each $i \geq 1$. Hence for each such branch there exists $x^* \in B_{X^*}$ such that $x^*(z_i) > (1 - \zeta)/2$ for every $i \geq 1$ so that $(z_i)_1^\infty$ is an ℓ_1^+ sequence. Now, for all $\alpha < \omega_1$, since S_3 is isomorphic to τ_ω , it follows that we may find a subtree of S_3 which is a weakly null tree isomorphic to $T(\alpha, s)$ and so $I_w^+(X) = \omega_1$ as claimed. \square

Remark 3.15. The condition that $\ell_1 \not\rightarrow X$ is necessary for (ix) above. Indeed, $I_w^+(\ell_1) = 0$ since there are no non-empty weakly null trees on ℓ_1 .

We now present the relationship between Rosenthal's wide-(s) and wide-(c) sequences and our ℓ_1^+ and ℓ_∞^+ sequences, in order to prove part (vi) of the above theorem. We first note the following result on the relationship between wide-(s) and wide-(c) sequences:

Proposition 3.16 (Rosenthal [Ro2]). *Let (b_j) be a sequence in X , finite or infinite, and let (e_j) be its difference sequence, so that $e_1 = b_1$ and $e_j = b_j - b_{j-1}$ ($j > 1$). Then for every $\lambda \geq 1$ there exists $\mu \geq 1$ such that*

- (i) *if (b_j) is λ -wide-(s), then (e_j) is μ -wide-(c);*
- (ii) *if (e_j) is λ -wide-(c), then (b_j) is μ -wide-(s).*

In the next lemma we show how to move between ℓ_1^+ and wide-(s) sequences, and between ℓ_∞^+ and wide-(c) sequences, and then in the lemma following we show how one can perturb this process and still keep control of the constants. This will be important when moving between these sequences in trees. We leave the proofs as exercises.

Lemma 3.17. *Let $\lambda, K \geq 1$, and let $(x_i)_1^m \subset X$.*

- (i) *If $(x_i)_1^m$ is an ℓ_1^+ - K sequence and $f \in S_{X^*}$ is such that $f(x_i) \geq 1/K$ for $1 \leq i \leq m$, then $(x_i/f(x_i))_1^m$ is a $2K$ -wide-(s) sequence.*
- (ii) *If $(x_i)_1^m$ is a λ -wide-(s) sequence, then $(x_i/\|x_i\|)_1^m$ is an ℓ_1^+ sequence with constant $\max(2\lambda, \lambda^2)$.*
- (iii) *If $(x_i)_1^m$ is an ℓ_∞^+ - K sequence and $(b_i)_1^m \subset [1/K, 1]$ is such that $\|\sum_1^m b_i x_i\| \leq K$, then $(b_i x_i)_1^m$ is a K -wide-(c) sequence.*
- (iv) *If $(x_i)_1^m$ is a λ -wide-(c) sequence, then $(x_i/\|x_i\|)_1^m$ is an ℓ_∞^+ - λ sequence.*

Lemma 3.18. *Let $K \geq 1$ and $\varepsilon > 0$. Then there exist $\varepsilon_i \searrow 0$ such that for $(x_i)_1^m \subset X$,*

- (i) if $(x_i)_1^m$ is an ℓ_1^+ - K sequence, $f \in S_{X^*}$ satisfies $f(x_i) \geq 1/K$ for $i \leq m$, and $(c_i)_1^m \subset [1/K, 1]$ is chosen so that $|f(x_i) - c_i| < \varepsilon_i$ for $i \leq m$, then $(x_i/c_i)_1^m$ is $2(K + \varepsilon)$ -wide-(s);
- (ii) if $(x_i)_1^m$ is an ℓ_∞^+ - K sequence, $(b_i)_1^m \subset [1/K, 1]$ satisfies $\|\sum_1^m b_i x_i\| \leq K$, and $(c_i)_1^m \subset [1/K, 1]$ is chosen so that $|b_i - c_i| < \varepsilon_i$ for $i \leq m$, then $(c_i x_i)_1^m$ is $(K + \varepsilon)$ -wide-(c).

The next lemma gives the framework for applying the previous two results to swap between whole trees of these sequences.

Lemma 3.19. *Let $\alpha < \omega_1$, $\delta > 0$ and $(\varepsilon_i)_1^\infty \subset \mathbf{R}^+$. Let T be a tree of order α , $(z^i)_1^\infty$ the sequence of terminal nodes of T , with $z^i = (x_j^i)_{j=1}^{m_i}$, and for each i , let f^i be a map from $\{x_1^i, \dots, x_{m_i}^i\}$ into $[\delta, 1]$. Then there exists an increasing sequence $N \subseteq \mathbf{N}$ such that the subtree $\bar{T} = \{z \in T : z \leq z^n \text{ for some } n \in N\}$ of T has order α and $|f^n(x_l^n) - f^{n'}(x_l^{n'})| < \varepsilon_l$ whenever $n, n' \in N$ and $x_k^n = x_k^{n'}$ for $1 \leq k \leq l$.*

Proof. We prove this by induction on α . There is nothing to prove for $\alpha \leq \omega$, and if the result has been proven for every $\beta < \alpha$, then it is also clear when α is a limit ordinal. Thus suppose the result has been proven for α' , let $\alpha = \alpha' + 1$, and let T be a tree of order α . By taking a subtree we may assume that T has a unique initial node, $z = (x_1)$, and let $S = \{(x_j)_2^m : (x_j)_1^m \in T\}$, a tree of order α' . The terminal nodes of S are $w^i = (x_j^i)_{j=2}^{m_i}$, and let $N \subseteq \mathbf{N}$ be the sequence from the induction hypothesis on S for α', δ and the sequence $(\varepsilon_i)_{i=2}^\infty$. Now let T' be the subtree of T , $\{z \in T : z \leq z^n \text{ for some } n \in N\}$. We have that $|f^n(x_l^n) - f^{n'}(x_l^{n'})| < \varepsilon_l$ whenever $n, n' \in N$, $l \geq 2$ and $x_k^n = x_k^{n'}$ for $2 \leq k \leq l$. We must now stabilize the maps on $x_1 = x_1^n$ for each $n \in N$.

Let $m = \lceil (1 - \delta)/\varepsilon_1 \rceil + 1$ and for $1 \leq r \leq m$ let

$$M_r = \{n \in N : f^n(x_1) \in [\delta + (r - 1)\varepsilon_1, \delta + r\varepsilon_1]\}.$$

This forms a partition of the terminal nodes of T' and by [JO] Lemma 5.10 one of the trees

$$T'(M_r) = \{y \in T' : y \leq z^n \text{ for some } n \in M_r\}$$

has order α for some $1 \leq r_0 \leq m$. The sequence $M_{r_0} \subseteq N \subseteq \mathbf{N}$ is now the required sequence. \square

Proof of Theorem 3.14 (vi). We shall show that for each $\alpha < \omega_1$ there exists an ℓ_1^+ -tree on X of order α if and only if there exists an ℓ_∞^+ -tree on X of order α .

Let $\alpha < \omega_1$ and let T be an ℓ_1^+ - K -tree on X of order α for some $K \geq 1$. Choose $0 < \varepsilon \ll 1/K$ and a sequence $(\varepsilon_i)_1^\infty \subset (0, \varepsilon)$ decreasing rapidly to zero. Let \bar{T} be the tree obtained when Lemma 3.19 is applied to T with $\delta = 1/K$ where the functions f^i on T are in S_{X^*} with $f^i(x_j^i) \geq 1/K$ ($1 \leq j \leq m_i$) for each terminal node $(x_j^i)_{j=1}^{m_i}$. We can find these functions since the terminal nodes are ℓ_1^+ - K

sequences. Let $f : \{x_j^i : 1 \leq j \leq m_i, i = 1, 2, \dots\} \rightarrow [1/K, 1]$ be the map $f(x_j^i) = f^k(x_j^i)$ where $k = \min\{i' \geq 1 : x_j^{i'} = x_j^i\}$ and let

$$S = \bigcup_{i=1}^{\infty} \left\{ \left(\frac{x_1^i}{f(x_1^i)} \right), \dots, \left(\frac{x_1^i}{f(x_1^i)}, \dots, \frac{x_{m_i}^i}{f(x_{m_i}^i)} \right) \right\},$$

Clearly S has order α and is a $2(K + \varepsilon)$ -wide-(s) tree by Lemma 3.18.

Now let $U = \{(y_1, y_2 - y_1, \dots, y_n - y_{n-1}) : (y_j)_1^n \in S\}$, then U is a wide-(c) tree by Proposition 3.16 above and clearly U is isomorphic to S so that $o(U) = \alpha$. Finally let $V = \{(u_i / \|u_i\|_1^m) : (u_i)_1^m \in U\}$, then V is an ℓ_∞^+ -tree by Lemma 3.17 (iv) and isomorphic to U so it is of order α . This completes the proof of one implication. The other is similar. \square

Remark 3.20. Notice that this proof also shows that the wide-(s) and wide-(c) indices are both equal to $I^+(X)$.

In Theorem 1.1 of [JO] the following extension of the finite version of the result of James [Ja1] that ℓ_1 is not distortable is shown. Given $K \geq 1$, $\varepsilon > 0$ and $\alpha < \omega_1$ there exists $\beta < \omega_1$ such that if T is an ℓ_1 - K -tree on X of order β , then there exists a block tree T' of T which is an ℓ_1 -tree with constant $1 + \varepsilon$ and order α . We have the analogous results for ℓ_1^+ -block basis trees and for ℓ_1^+ -weakly null trees:

Theorem 3.21. *Let $K \geq 1$, $\varepsilon > 0$ and $\alpha < \omega_1$.*

- (i) *There exists $\beta < \omega_1$ such that for every ℓ_1^+ - K -weakly null tree T of order β (i.e. isomorphic to $T(\beta, s)$) on a Banach space with separable dual there exists a block tree T' of T which is an ℓ_1^+ -($1 + \varepsilon$)-weakly null tree of order α .*
- (ii) *There exists $\gamma < \omega_1$ such that for every ℓ_1^+ - K -block basis tree T of order γ on a Banach space with a shrinking basis there exists a block tree T' of T which is an ℓ_1^+ -($1 + \varepsilon$)-block basis tree of order α .*

This theorem is slightly harder to prove than Theorem 1.1 of [JO] because not only do we have to reduce the ℓ_1^+ estimate of the nodes, but we also have to reduce the basis constant to $1 + \varepsilon$. In the proof of part (i), we start off with a weakly null tree of order β . We prune the tree using the Pruning Lemma so that each s -subsequence and each branch is a weakly null basic sequence with basis constant $(1 + \varepsilon)$. One may then follow the same argument as in the proof of [JO] Theorem 1.1 to reduce an ℓ_1^+ - K^2 -weakly null tree isomorphic to $T(\alpha^2, s)$ to an ℓ_1^+ - K -weakly null tree isomorphic to $T(\alpha, s)$, and then complete the proof using the method of James [Ja1] as in the last part of the

proof of [JO] Theorem 1.1. Care must be taken to preserve the weakly null structure, but one can achieve this since X^* is separable and the original trees are weakly null.

For part (ii) of the theorem we take a block basis tree of order $\gamma = \omega \cdot \beta$ which we may assume is isomorphic to the minimal tree $T(\beta, \omega)$. From this we can extract an ℓ_1^+ - K -block basis tree which is isomorphic to $T(\beta, s)$ with the property that the s -subsequences are block bases. But now the s -subsequences are weakly null because the basis is shrinking and we may prune the tree to obtain a tree still isomorphic to $T(\beta, s)$ whose nodes are ℓ_1^+ - K block bases and $(1 + \varepsilon)$ basic. We then follow the same argument as in [JO] to obtain the result.

Theorem 3.22. *If X is a Banach space with separable dual, then $I_w^+(X) = \omega^\alpha$ for some $\alpha < \omega_1$.*

Proof. From Theorem 3.14 (ix) we know that $I_w^+(X) < \omega_1$, and so it suffices to show (see e.g. Monk [Mo]) that if $\beta < I_w^+(X)$, then $\beta \cdot 2 < I_w^+(X)$. We may regard X as a subspace of $C[0, 1]$, and let $(e_i)_1^\infty$ be a monotone basis for $C[0, 1]$. Let T be an ℓ_1^+ - K -weakly null tree on X of order β . To make a tree of order $\beta \cdot 2$ we want to add a tree of order β after each terminal node of T .

Let a sequence $(x_i)_1^\infty$ in X have property $\mathbf{Q}(\varepsilon)$ for $\varepsilon > 0$ if it is normalized and weakly null. Let $(x_i)_1^\infty$ have property $\mathbf{P}(\varepsilon)$ if it is an ε perturbation of a normalized block basis of $(e_i)_1^\infty$. Properties $\mathbf{P}(\varepsilon)$ and $\mathbf{Q}(\varepsilon)$ clearly satisfy conditions PL(1)–(3) of the Pruning Lemma. Note also that if we apply the Pruning Lemma to a tree T for $\varepsilon > 0$ and a sequence $(u_i)_1^l$ satisfying $\mathbf{P}(\varepsilon)$, then the resulting sequences $(u_1, \dots, u_l, x_1, x_2, \dots)$ are $(1 + \varepsilon)/(1 - \varepsilon)$ -basic.

We apply the Pruning Lemma to T with $\varepsilon < \min\{1/6, 1/(4K)\}$ and the empty sequence, so we may assume that for every s -node $(z^i)_1^\infty$ of T with $z^i = (x_1, \dots, x_k, y_i)$, the sequence $(x_1, \dots, x_k, y_1, y_2, \dots)$ is normalized, weakly null and an ε perturbation of a normalized block basis of $(e_i)_1^\infty$.

Now let $(z^i)_1^\infty$ be the sequence of terminal nodes of T with $z^i = (x_j^i)_{j=1}^{k_i}$, and apply the Pruning Lemma to T for $(x_j^i)_{j=1}^{k_i}$ with $\varepsilon < \min\{1/6, 1/(4K)\}$ and $\delta = \varepsilon$, for each $i \geq 1$, to obtain a tree S_i which has $\mathbf{P}(2\varepsilon)$ for the sequence $(x_j^i)_{j=1}^{k_i}$.

To complete the proof we put the trees together as follows. Let

$$S = \bigcup_{i=1}^{\infty} \{(x_1^i), (x_1^i, x_2^i), \dots, (x_j^i)_{j=1}^{k_i}, (x_1^i, \dots, x_{k_i}^i, y_1, \dots, y_l) : (y_j)_1^l \in S_i\}.$$

The ordering on S is that inherited from T and the trees S_i . It is easy to see that S is an ℓ_1^+ -weakly null tree. Indeed, if $z = (x_1, \dots, x_k, y_1, \dots, y_l) \in S$, then the sequence is 2-basic and both $(x_i)_1^k$ and $(y_j)_1^l$ are ℓ_1^+ - K sequences so that $(x_1, \dots, x_k, y_1, \dots, y_l)$ is an ℓ_1^+ - $6K$ sequence. Since

we have extended only the terminal nodes it is clear that the new tree is weakly null. Finally, $o(S) = \beta \cdot 2$, since $o(S_i) = \beta$, and hence $(S)^\beta = T$. \square

Theorem 3.23. *Let X be a Banach space with shrinking basis $(e_i)_1^\infty$. If $I_w^+(X) \geq \omega^\omega$, then $I_w^+(X) = I_b^+(X)$, otherwise, if $I_w^+(X) = \omega^n$, then $I_b^+(X) = \omega^n$ or ω^{n+1} .*

Proof. We first show by induction on α that if T is an ℓ_1^+ -block basis tree on X of order $\omega \cdot \alpha$, then we can extract a certain subtree S isomorphic to $T(\alpha, s)$. The tree S will have the property that each s -subsequence $(x_i)_1^\infty$ is a normalized block basis of the shrinking basis $(e_i)_1^\infty$ and hence is weakly null. Thus S will be an ℓ_1^+ -weakly null tree, and so $I_b^+(X) \leq \omega \cdot I_w^+(X)$. We prove this by induction on α .

For $\alpha = 1$ we may assume that $T \simeq T_\omega \simeq T(1, \omega)$ so that T consists of disjoint branches b_n of length at least n for each $n \geq 1$. Since each branch is a block basis of $(e_i)_1^\infty$, there must exist a sequence $(n_i)_1^\infty$ and a vector x_i in one of the nodes of branch b_{n_i} such that $(x_i)_1^\infty$ is a normalized block basis of $(e_i)_1^\infty$. Set $z^i = (x_i)$ for each i , then $T' = \{z^i : i \geq 1\}$, a sequence of incomparable nodes, is the required subtree.

If the result has been proven for α , then we let T be an ℓ_1^+ -block basis tree with order $\omega \cdot (\alpha + 1)$ and assume $T \simeq T(\alpha + 1, \omega)$, since this is a minimal tree of order $\omega \cdot (\alpha + 1)$. Recall that $T(\alpha + 1, \omega)$ is constructed by taking T_ω , and then after each terminal node putting a tree isomorphic to $T(\alpha, \omega)$. Applying the above argument for $\alpha = 1$ to the initial part of the tree, which is isomorphic to T_ω , we may construct a sequence of incomparable nodes $(z^i)_1^\infty$, with $z^i = (x_i)$ and $(x_i)_1^\infty$ a normalized block basis of $(e_i)_1^\infty$. After each node z^i we have a tree isomorphic to $T(\alpha, \omega)$ from which we may construct an ℓ_1^+ -weakly null subtree of order α with the required properties, using the induction hypothesis. Putting these trees together with the nodes $(z^i)_1^\infty$ we obtain the desired ℓ_1^+ -weakly null tree.

If α is a limit ordinal and the result has been proven for any ordinal smaller than α , then let T be an ℓ_1^+ -block basis tree on X isomorphic to the minimal tree $T(\alpha, \omega)$. Recall that $T(\alpha, \omega)$ is constructed by taking a certain sequence of ordinals $(\alpha_n)_1^\infty$ increasing to α and letting $T(\alpha, \omega)$ be the disjoint union of the trees S_n isomorphic to $T(\alpha_n, \omega)$. By the induction hypothesis we may find a subtree \bar{S}_n of S_n for each n which is isomorphic to $T(\alpha_n, s)$ and has the required properties. We must now be a little careful when putting the trees \bar{S}_n together. We cannot just take their union since we will need the initial nodes to form a weakly null sequence. Let $(z^{n,i})_{i=1}^\infty$ be the sequence of initial nodes of \bar{S}_n , with $z^{n,i} = (x_{n,i})$. Since for each n the sequence $(x_{n,i})_{i=1}^\infty$ is a normalized block basis of $(e_i)_1^\infty$, it follows that we may find a sequence $(x_i)_1^\infty \subset \{x_{n,i} : n, i \geq 1\}$ and $1 \leq k_1 < k_2 < \dots$ such that $(x_i)_1^\infty$ is a normalized block basis of $(e_i)_1^\infty$ and $x_i = x_{\varphi(i), k_i}$. Let

$z^i = (x_i) = z^{\varphi(i), k_i}$ and set $S'_n = \{x \in \bar{S}_n : x \geq z^i, \text{ for some } i \in \varphi^{-1}(n)\}$. Then $T' = \cup_n S'_n$ is the required tree. This completes the first part of the proof and shows that $I_b^+(X) \leq \omega \cdot I_w^+(X)$.

We noted in the proof of Theorem 3.14 (ix) that $I_w^+(X) \leq I_b^+(X)$, and since we know that both indices are of the form ω^α for some $\alpha < \omega_1$ the result follows from the two inequalities. \square

4. THE SZLENK INDEX

In this section we examine the Szlenk index, another isomorphic invariant of a Banach space, introduced by Szlenk [Sz]. This is calculated in a different way to the ℓ_1 -indices; it uses collections of subsets in the dual ball, indexed by countable ordinals. We show that the Szlenk index is in fact the same as the ℓ_1^+ -weakly null index provided the space does not contain ℓ_1 .

Definition 4.1. For a fixed $\varepsilon > 0$ we construct inductively sets $P_\alpha(\varepsilon) \subseteq B_{X^*}$. Let $P_0(\varepsilon) = B_{X^*}$ and if we have constructed $P_\alpha(\varepsilon)$, then let

$$P_{\alpha+1}(\varepsilon) = \{f \in B_{X^*} : \exists (f_m)_1^\infty \subset P_\alpha(\varepsilon) \text{ with } f_m \xrightarrow{w^*} f \text{ and } \liminf \|f_m - f\| \geq \varepsilon\}.$$

If α is a limit ordinal and we have chosen $P_\beta(\varepsilon)$ for each $\beta < \alpha$, then let

$$P_\alpha(\varepsilon) = \bigcap_{\beta < \alpha} P_\beta(\varepsilon).$$

We define the ε -Szlenk index of X to be

$$\eta(\varepsilon, X) = \sup\{\alpha : P_\alpha(\varepsilon) \neq \emptyset\},$$

if such an α exists, and ω_1 otherwise, and the Szlenk index of a Banach space X as

$$\eta(X) = \sup_{\varepsilon > 0} \eta(\varepsilon, X).$$

One can show that if ℓ_1 does not embed into X , then $\eta(X) < \omega_1$ if and only if X^* is separable. Indeed, if X^* is not separable, then by Stegall's result used above [St] there exists a homeomorphic copy of Δ in (B_{X^*}, w^*) which is $(1-\varepsilon)$ -separated, i.e. $\|x^* - y^*\| > 1-\varepsilon$ for $x^*, y^* \in \Delta$ with $x^* \neq y^*$. Thus $\Delta \subseteq P_\alpha(1/2)$ for each $\alpha < \omega_1$.

In his original definition Szlenk used sets $P'_\alpha(\varepsilon)$ defined in a similar manner to the sets $P_\alpha(\varepsilon)$ above, except at successor ordinals he had

$$P'_{\alpha+1}(\varepsilon) = \{f \in X^* : \exists (x_m)_1^\infty \subset B_X, (f_m) \subset P'_\alpha(\varepsilon)$$

$$\text{such that } f_m \xrightarrow{w^*} f, x_m \xrightarrow{w} 0 \text{ and } \limsup |f_m(x_m)| \geq \varepsilon\}.$$

The original Szlenk indices $\eta'(\varepsilon, X)$ and $\eta'(X)$ were defined as before, but using the sets $P'_\alpha(\varepsilon)$. Szlenk then showed that if X^* is separable, then $\eta'(X) < \omega_1$. These two definitions may give different values for the ε -indices. However, we have using Rosenthal's ℓ_1 theorem that if X does not contain ℓ_1 , then

$$P'_\alpha(\varepsilon) \subseteq P_\alpha(\varepsilon) \subseteq P'_\alpha(\varepsilon/2)$$

and hence

$$\eta'(\varepsilon, X) \leq \eta(\varepsilon, X) \leq \eta'(\varepsilon/2, X) .$$

Thus, if $\ell_1 \not\hookrightarrow X$, then $\eta'(X) = \eta(X)$. Since we shall only be considering spaces which do not contain ℓ_1 , in the sequel we shall apply the definition for the Szlenk index using the sets $P_\alpha(\varepsilon)$.

Theorem 4.2. *If X is a separable Banach space not containing ℓ_1 , then $\eta(X) = I_w^+(X)$.*

This result shows that despite the Szlenk index being calculated using subsets of the unit ball of X^* , while the ℓ_1^+ -weakly null index is calculated using trees on the unit ball of X , the two indices are in fact the same. Moreover we show that from the sets $P_\alpha(\varepsilon)$ used to calculate the Szlenk index one can generate trees which are analogous to the trees used in the ℓ_1^+ -weakly null index.

By Theorem 3.14 (ix) and the remarks on the Szlenk index above we have that if ℓ_1 does not embed inside X , $\eta(X) < \omega_1$ if and only if X^* is separable if and only if $I_w^+(X) < \omega_1$. Thus in proving the theorem we may restrict ourselves to the case where X^* is separable. The proof is in two parts. In the first we show that if $P_\alpha(\varepsilon) \neq \emptyset$, then there exists an ℓ_1^+ -weakly null tree on X of order α with constant $8/\varepsilon$. In the second part we demonstrate that if we have an ℓ_1^+ -weakly null tree on X with constant K and order α , then $P_\alpha(1/K) \neq \emptyset$. Thus our first task is to prove

Proposition 4.3. *If $P_\alpha(\varepsilon) \neq \emptyset$, then there exists an ℓ_1^+ -weakly null tree on X of order α with constant $8/\varepsilon$.*

To prove this proposition we first construct a tree of order α isomorphic to $T(\alpha, s)$ on B_{X^*} . From this tree we construct an isomorphic tree on B_X which is an ℓ_1^+ -weakly null tree. We construct the tree on B_{X^*} in the next two lemmas, and in Lemma 4.6 describe the properties $\mathbf{P}(\varepsilon)$ and $\mathbf{Q}(\varepsilon)$ needed to construct the tree on B_X from the tree on B_{X^*} .

Lemma 4.4. *If $P_\alpha(\varepsilon) \neq \emptyset$, then for each $f_0 \in P_\alpha(\varepsilon)$ and each weak* relatively open neighborhood O of f_0 , with respect to $P_\alpha(\varepsilon)$, there exists a tree T on O , isomorphic to $T(\alpha, s)$, such that if $z = (f_i)_1^k \in T$ has immediate successors $(z^j)_1^\infty$ with $z^j = (f_1, \dots, f_k, g_j)$, then $g_j \xrightarrow{w^*} f_k$ as $(j \rightarrow \infty)$*

and $\liminf_j \|g_j - f_k\| \geq \varepsilon$. Also, if $(z^j)_1^\infty$ is the sequence of initial nodes, with $z^j = (g_j)$, then $g_j \xrightarrow{w^*} f_0$ ($j \rightarrow \infty$) and $\liminf_j \|g_j - f_0\| \geq \varepsilon$.

Proof of Lemma 4.4. As usual we use induction on α . For the initial case $\alpha = 1$, let $f_0 \in P_1(\varepsilon)$ and let O be a weak* relatively open neighborhood of f_0 . We may find $(g_j)_1^\infty \subset O$ such that $g_j \xrightarrow{w^*} f_0$ and $\liminf \|g_j - f_0\| \geq \varepsilon$. Set $z^j = (g_j)$, then $T = \{z^j : j \geq 1\}$ is the required tree.

We next suppose the result has been proven for α ; let $f_0 \in P_{\alpha+1}(\varepsilon)$, and let O be a weak* relatively open neighborhood of f_0 . We may find $(g_j)_1^\infty \subset O$ with $g_j \xrightarrow{w^*} f_0$ and $\liminf \|g_j - f_0\| \geq \varepsilon$. From the induction hypothesis, for each $j \geq 1$ there exists a tree S_j isomorphic to $T(\alpha, s)$ satisfying the requirements for $g_j \in P_\alpha(\varepsilon)$. Let $S'_j = \{(g_j, h_1, \dots, h_k) : (h_i)_1^k \in S_j\}$ so that the trees S'_j are disjoint. Define $T = \cup_j S'_j$, then $T \simeq T(\alpha + 1, s)$ and satisfies the requirements of the lemma for f_0 .

If α is a limit ordinal and the result has been proven for each $\beta < \alpha$, let $(\alpha_n)_1^\infty$ be the sequence of successor ordinals increasing to α so that $T(\alpha, s) = \cup_n T(\alpha_n, s)$. Let $f_0 \in P_\alpha(\varepsilon)$ (so that $f_0 \in P_{\alpha_n}(\varepsilon)$ for each $n \geq 1$) and let $O \supseteq O_1 \supseteq O_2 \supseteq \dots$ be a decreasing collection of weak* relatively open neighborhoods of f_0 , so that $\cap_i O_i = \{f_0\}$, which may be chosen since X is separable. By the induction hypothesis we may find a tree S_n isomorphic to $T(\alpha_n, s)$ satisfying the lemma for f_0 and O_n for each $n \geq 1$. Let $(z^{n,i})_{i=1}^\infty$ be the sequence of initial nodes of S_n with $z^{n,i} = (f_{n,i})$, so that $f_{n,i} \xrightarrow{w^*} f_0$ ($i \rightarrow \infty$) and $f_{n,i} \in O_n$ for each $i \geq 1$ and every $n \geq 1$. Since the sets O_n are decreasing we may find a subsequence $(f_i)_1^\infty$ of $\{f_{n,i} : n, i \geq 1\}$ and numbers $1 \leq k_1 < k_2 < \dots$ with $f_i = f_{\varphi(i), k_i}$ for each i (where φ is the function from the Pruning Lemma) such that $f_i \xrightarrow{w^*} f_0$ and $\liminf \|f_i - f_0\| \geq \varepsilon$. It is at this limit ordinal stage that we use the relatively open neighborhoods O_n to ensure that we choose $f_i \in S_i$, so that the order of the tree T below will be α . The tree

$$T = \bigcup_n \{z \in S_n : z \geq f_{i, k_i}, i \in \varphi^{-1}(n)\}$$

satisfies the requirements of the lemma. \square

Lemma 4.5. *If $P_\alpha(\varepsilon) \neq \emptyset$, then for any $\delta > 0$ there exists a tree T isomorphic to $T(\alpha, s)$ on B_{X^*} such that*

- (i) $h_j \xrightarrow{w^*} 0$ and $\varepsilon/2 - \delta \leq \|h_j\| \leq 1$ ($j \geq 1$) for every s -subsequence $(h_j)_1^\infty$ of T ;
- (ii) $\|\sum_k^l g_i\| \leq 1$ ($1 \leq k \leq l \leq m$) for every $(g_i)_1^m \in T$.

Proof. Let $f_0 \in P_\alpha(\varepsilon)$ and let T be the tree for f_0 from the previous lemma for $O = P_\alpha(\varepsilon)$. Replace each node $z = (f_i)_1^m \in T$ with the node

$$\bar{z} = (\tfrac{1}{2}(f_1 - f_0), \tfrac{1}{2}(f_2 - f_1), \dots, \tfrac{1}{2}(f_m - f_{m-1}))$$

to obtain the tree \bar{T} which is still isomorphic to $T(\alpha, s)$. Clearly, if $(h_j)_1^\infty$ is any s -subsequence, then $h_j \xrightarrow{w^*} 0$ and $\liminf_j \|h_j\| \geq \varepsilon/2$. Let $(x_i^*)_1^\infty$ have property $\mathbf{Q}(\varepsilon)$ if it is weak* null with $\liminf_j \|x_j^*\| \geq \varepsilon/2$, and property $\mathbf{P}(\varepsilon)$ if it is weak* null with $\|x_j^*\| \geq \varepsilon/2$ for every i . It is clear that $\mathbf{Q}(\varepsilon)$ and $\mathbf{P}(\varepsilon)$ satisfy condition PL(1) of the Pruning Lemma, as modified by Remark 2.10 (iii) after it and we obtain condition PL(2) using the fact that X is separable. Thus we may apply the Pruning Lemma and prune \bar{T} to obtain a tree T' with property $\mathbf{P}(\varepsilon - \delta)$ satisfying condition (i) above. To see that (ii) holds, note that each node $z = (g_i)_1^m \in T$ is of the form $(\frac{1}{2}(f_1 - f_0), \frac{1}{2}(f_2 - f_1), \dots, \frac{1}{2}(f_m - f_{m-1}))$ so that $\|\sum_k^l g_i\| = \frac{1}{2}\|f_l - f_{k-1}\| \leq 1$. \square

Our next lemma contains the basic relationship between the weak* null trees constructed above and trees in X . The lemma is stated so as to verify the hypotheses of the Pruning Lemma with the additional conditions of Remark 2.10 (iv).

Lemma 4.6. *Let X be a Banach space with separable dual, let $\varepsilon > 0$ and let $(f_i)_1^\infty \subseteq B_{X^*}$ be weak* null with $\varepsilon/2 \leq \|f_i\| \leq 1$ for every $i \geq 1$. Then there exists a subsequence $(f'_i)_1^\infty$ of $(f_i)_1^\infty$ and a weakly null sequence $(x_i)_1^\infty \subseteq S_X$ such that $f'_i(x_i) \geq \varepsilon/5$, $|f'_i(x_j)| < \varepsilon/2^{i+6}$ whenever $i \neq j$ and $(x_i)_1^k$ is $(1 + \varepsilon(1 - 2^{-k}))$ basic for every $k \geq 1$.*

Proof. Let $0 < \delta < \varepsilon$, to be chosen later. We first choose a sequence $(y_i)_1^\infty \subset S_X$ with $f_i(y_i) \geq \varepsilon/2 - \delta$ for each i . Since X^* is separable we may assume $(y_i)_1^\infty$ is weakly Cauchy (by taking a subsequence of $(y_i)_1^\infty$ and then the same subsequence of $(f_i)_1^\infty$). Again, by taking subsequences and using that $f_i \xrightarrow{w^*} 0$, we may assume that $|f_n(y_i)| < \theta(\varepsilon, n)$ if $i < n$ (where $\theta(\varepsilon, n)$ is small, to be chosen later). Now set

$$x_n = \frac{y_n - y_{n-1}}{\|y_n - y_{n-1}\|},$$

so that, since $f_n(y_n - y_{n-1}) \geq \varepsilon/2 - \delta - \theta(\varepsilon, n)$, it follows that $\|y_n - y_{n-1}\| \geq \varepsilon/2 - \delta - \theta(\varepsilon, n)$, and hence $x_n \xrightarrow{w} 0$ and $f_n(x_n) \geq \varepsilon/4 - \delta/2 - \theta(\varepsilon, n)/2$. Further, for $i < n$,

$$|f_n(x_i)| = \frac{|f_n(y_i) - f_n(y_{i-1})|}{\|y_i - y_{i-1}\|} < \frac{2\theta(\varepsilon, n)}{\varepsilon/2 - \delta - \theta(\varepsilon, n)}.$$

If $\delta < \varepsilon/40$ and $\theta(\varepsilon, n) = \varepsilon^2/2^{i+10}$, then $f_i(x_i) \geq \varepsilon/5$ and $|f_n(x_i)| < \varepsilon/2^{n+6}$ when $i < n$. Next, since $x_n \xrightarrow{w} 0$, we may pass to subsequences of $(f_i)_1^\infty$ and $(x_i)_1^\infty$ to obtain $|f_i(x_j)| < \varepsilon/2^{i+6}$ when $i \neq j$. We now pass to one last pair of subsequences $(f'_i)_1^\infty$ and $(x'_i)_1^\infty$ so that $(x'_i)_1^k$ is $(1 + \varepsilon(1 - 2^{-k}))$ basic for every $k \geq 1$ as required. \square

Proof of Proposition 4.3. We have that $P_\alpha(\varepsilon) \neq \emptyset$ and we want to construct an ℓ_1^+ -weakly null tree on X of order α and constant $K = K(\varepsilon) = 8/\varepsilon$. Let T be the tree on B_{X^*} for some

$\delta < \varepsilon/12$ from Lemma 4.5. We want to construct a tree S in S_X , isomorphic to T , so that if $(f_i)_1^m \in T$ has immediate successors $(f_1, \dots, f_m, f_{m+1}), (f_1, \dots, f_m, f_{m+2}), \dots$ etc., and $(x_i)_1^m, (x_1, \dots, x_m, x_{m+1}), (x_1, \dots, x_m, x_{m+2}), \dots$ are the corresponding nodes of S , then $(x_i)_1^\infty$ is weakly null, $f_i(x_i) \geq \varepsilon/5$, $|f_i(x_j)| < \varepsilon/2^{i+6}$ whenever $i \neq j$ and $(x_i)_1^k$ is $(1 + \varepsilon(1 - 2^{-k}))$ basic for every $k \geq 1$. The proof is very similar to that of the Pruning Lemma, although a little stronger as we must keep track of two trees S and T , so we will not give it here.

We claim that S is the required ℓ_1^+ - K -weakly null tree. We already know that S is a weakly null tree. We must show that if $(x_i)_1^m \in S$, then $(x_i)_1^m$ is an ℓ_1^+ - K -sequence. We know that $(x_i)_1^m$ is $(1 + \varepsilon)$ basic, so we seek $f \in S_{X^*}$ such that $f(x_i) \geq 8/\varepsilon$ for $1 \leq i \leq m$. Let $(f_i)_1^m$ be the corresponding node in T to $(x_i)_1^m$ and recall that $\|\sum_1^m f_i\| \leq 1$. Now,

$$\sum_{j=1}^m f_j(x_i) = f_i(x_i) + \sum_{j \neq i} f_j(x_i) \geq f_i(x_i) - \sum_{j \neq i} |f_j(x_i)| \geq \frac{\varepsilon}{6} - \sum_{j \neq i} 2^{-j-6}\varepsilon \geq \frac{\varepsilon}{8},$$

for $1 \leq i \leq m$. Finally, setting $f = \sum_{j=1}^m f_j / \|\sum_{j=1}^m f_j\|$ we still have $f(x_i) \geq \varepsilon/8$ for each i , and hence $(x_i)_1^m$ is an ℓ_1^+ - K -sequence with $K = 8/\varepsilon$. \square

Remark 4.7. One can be more careful with the estimates in the proofs of Lemma 4.6 and Proposition 4.3, and obtain an ℓ_1^+ - K -weakly null tree on X of order α with $1/K = \varepsilon/4 - \delta$ for any $\delta > 0$.

This completes the first part of the proof of Theorem 4.2. We now have to show how to get from an ℓ_1^+ -weakly null tree on B_X to the sets required in the calculation of the Szlenk index.

Definition 4.8. If T is an ℓ_1^+ - K -tree on X and $(x_i)_1^n$ is a terminal node of T , then let $\gamma = \{y \in T : y \leq (x_i)_1^n\}$ be the branch of T ending at $(x_i)_1^n$. A K -branch functional of γ is an element $f_\gamma \in KB_{X^*}$ with $f_\gamma(x_i) \geq 1$ for each i . These exist from the equivalent formulation of ℓ_1^+ sequences in Fact 3.7. A full set of branch functionals of T is a subset of X^* which contains a branch functional for each branch of T .

Lemma 4.9. If T is an ℓ_1^+ - K -weakly null tree of order α and W is a weak* closed subset of KB_{X^*} which contains a full set of K -branch functionals of T , then $W \cap KP_\alpha(1/K) \neq \emptyset$.

Proof. As usual we proceed by induction on α . If $o(T) = 1$, then $T = \{z^i : i \geq 1\}$ where $z^i = (x_i)$ and $(x_i)_1^\infty$ is a normalized weakly null sequence. For each i pick $f_i \in W$ with $f_i(x_i) \geq 1$, then choose a subsequence $(f_{n_i})_1^\infty$ which converges weak* to some $f \in W$. Choose a sequence $\varepsilon_i \searrow 0$;

since $(x_i)_1^\infty$ is weakly null, it follows that for each i there exists $m_i \geq 1$ such that $|f(x_j)| < \varepsilon_j$ for every $j \geq m_i$. But now

$$|f_{n_j}(x_{n_j}) - f(x_{n_j})| \geq 1 - \varepsilon_i \text{ for every } j \geq m_i,$$

so that $\liminf \|f_{n_j} - f\| \geq 1$ and hence $f \in KP_1(1/K)$. The result for the case $\alpha = 1$ follows easily from this.

If the result has been proven for α , let T be an ℓ_1^+ - K -weakly null tree of order $\alpha + 1$ and let W be a weak* closed subset of KB_{X^*} which contains a full set of branch functionals of T . Let $(z^i)_1^\infty$ be the sequence of initial nodes of T with $z^i = (x_i)$ and $(x_i)_1^\infty$ a normalized weakly null sequence. For each i let W_i be the weak* closure of the set of branch functionals of T in W for branches whose initial node is z^i . Thus $f(x_i) \geq 1$ for every $f \in W_i$. Further, let $T_i = \{y \in T : y > z^i\}$, so that T_i is an ℓ_1^+ - K -weakly null tree of order α , and W_i is a weak* closed subset of KB_{X^*} which contains a full set of branch functionals of T_i . Hence, by the induction hypothesis, there exists $f_i \in W_i$ with $f_i \in W_i \cap KP_\alpha(1/K)$. Now, $f_i(x_i) \geq 1$ for each i , so we may now proceed as in the case $\alpha = 1$ to obtain $f_{n_i} \xrightarrow{w^*} f$ with $\liminf \|f_{n_i} - f\| \geq 1$. Thus $f \in KP_{\alpha+1}(1/K)$. Then, since W is weak* closed, and since $W_i \subseteq W$ for each i , it follows that $f \in W \cap KP_{\alpha+1}(1/K)$ as required.

For the case where α is a limit ordinal we simply note that if the result has been proven for each $\beta < \alpha$, and if we have T and W as in the statement of the lemma, then $W \cap KP_\beta(1/K) \neq \emptyset$ for each $\beta < \alpha$. This forms a countable decreasing sequence of non-empty weak* closed sets in the weak* compact set KB_{X^*} . Thus $W \cap KP_\alpha(1/K) = W \cap (\bigcap_{\beta < \alpha} KP_\beta(1/K)) = \bigcap_{\beta < \alpha} (W \cap KP_\beta(1/K)) \neq \emptyset$, which completes the proof. \square

Proposition 4.10. *If there exists an ℓ_1^+ - K -weakly null tree on X of order α , then $P_\alpha(1/K) \neq \emptyset$.*

Proof. If T is an ℓ_1^+ - K -weakly null tree on X of order α , then there exists a branch functional for each branch of T . Thus we may take $W = KB_{X^*}$ in the above lemma, to obtain $P_\alpha(1/K) = \frac{1}{K}W \cap P_\alpha(1/K) \neq \emptyset$. \square

5. THE ℓ_1 INDEX OF THE SCHREIER SPACES AND THE $C(\alpha)$ SPACES

In this section we calculate the ℓ_1 -indices of the Schreier spaces and the $C(\alpha)$ spaces using the results from the previous two sections. We first give some notation.

Definition 5.1 ([AA]). Let E, F be subsets of \mathbf{N} and $n \geq 1$. We write $E < F$ if F is empty or $\max E < \min F$; we write $n < E$ if $\{n\} < E$, and $n \leq E$ if $n = \min E$ or $n < E$. The Schreier

sets \mathcal{S}_α , for each $\alpha < \omega_1$, are defined inductively as follows: Let $\mathcal{S}_0 = \{\{n\} : n \geq 1\} \cup \{\emptyset\}$ and $\mathcal{S}_1 = \{F \subset \mathbf{N} : |F| \leq F\}$. (Note that this definition allows for $\emptyset \in \mathcal{S}_1$.) If \mathcal{S}_α has been defined, let

$$\mathcal{S}_{\alpha+1} = \{\cup_1^k F_i : k \leq F_1 < \dots < F_k, F_i \in \mathcal{S}_\alpha \ (i = 1, \dots, k), k \in \mathbf{N}\}.$$

If α is a limit ordinal with \mathcal{S}_β defined for each $\beta < \alpha$, choose and fix an increasing sequence of ordinals (α_n) with $\alpha = \sup_n \alpha_n$ and let

$$\mathcal{S}_\alpha = \bigcup_{n=1}^{\infty} \{F \in \mathcal{S}_{\alpha_n} : n \leq F\}.$$

Each \mathcal{S}_α has the following two important properties. First, if $F = \{m_1, \dots, m_k\} \in \mathcal{S}_\alpha$ and $n_1 < \dots < n_k$ satisfies: $m_i \leq n_i$ for $i \leq k$, then $\{n_1, \dots, n_k\} \in \mathcal{S}_\alpha$ (this is called *spreading*). Second, whenever $E \subset F$ and $F \in \mathcal{S}_\alpha$ then $E \in \mathcal{S}_\alpha$ (this is called *hereditary*).

For each $\alpha < \omega_1$ the Schreier set \mathcal{S}_α generates a tree, $\text{Tree}(\mathcal{S}_\alpha) = (\mathcal{S}_\alpha, \subseteq)$, ordered by inclusion. It is easy to see that the order of $\text{Tree}(\mathcal{S}_\alpha)$ is $\omega^\alpha + 1$ [AA].

Definition 5.2. The Schreier spaces generalize Schreier's example [Sch]; they were introduced in [AO] for α finite and in [AA] for α infinite. We first define c_{00} to be the linear space of all real sequences with finite support, and let $(e_i)_1^\infty$ be the unit vector basis of c_{00} . For each $\alpha < \omega_1$ let $\|\cdot\|_\alpha$ be the norm on c_{00} given by:

$$\left\| \sum a_i e_i \right\|_\alpha = \sup_{E \in \mathcal{S}_\alpha} \left| \sum_{i \in E} a_i \right|,$$

then the Schreier space X_α is the completion of $(c_{00}, \|\cdot\|_\alpha)$. Note that because \mathcal{S}_α is hereditary $(e_i)_1^\infty$ is a normalized 1-unconditional basis for X_α .

Definition 5.3. If $\alpha > 0$ is an ordinal, then $C(\alpha)$ denotes the Banach space of all continuous real-valued functions on the ordinals less than or equal to α , where $[1, \alpha] = \{\beta : \beta \leq \alpha\}$ has the order topology, with the norm $\|x\| = \sup_{\beta \in \alpha+1} |x(\beta)|$. Thus $C(\alpha) = C([1, \alpha])$ is the space of all continuous functions $x : [1, \alpha] \rightarrow \mathbf{R}$.

The following classical theorem of Bessaga and Pełczyński [BP] partitions the $C(\alpha)$ spaces ($\omega \leq \alpha < \omega_1$) into isomorphism classes.

Theorem 5.4 (Bessaga and Pełczyński). *Let $\omega \leq \alpha \leq \beta < \omega_1$, then $C(\alpha)$ is isomorphic to $C(\beta)$ if, and only if, $\beta < \alpha^\omega$. Furthermore, if we do have $\beta < \alpha^\omega$, then $C(\beta) \oplus C(\alpha)$ is isomorphic to $C(\alpha)$.*

Thus, in studying isomorphic invariants of the spaces $C(\alpha)$ for $\alpha < \omega_1$, and hence in particular when calculating the ℓ_1 -indices, it suffices to consider the spaces $C(\omega^{\omega^\beta})$ for $\beta < \omega_1$. It is well known (see [AB]) and not difficult to see that the Szlenk indices of the $C(\alpha)$ spaces are given by $\eta(C(\omega^{\omega^\alpha})) = \omega^{\alpha+1}$ and so by Theorem 4.2 $I_w^+(C(\omega^{\omega^\alpha})) = \omega^{\alpha+1}$.

The main result of this section is the following theorem.

Theorem 5.5. *For $1 \leq \alpha < \omega_1$*

- (i) $I(X_\alpha) = I_b(X_\alpha) = \omega^{\alpha+1}$ with respect to the unit vector basis $(e_i)_1^\infty$ of X_α ;
- (ii) $I_b(C(\omega^{\omega^\alpha})) = \omega^{\alpha+1}$ with respect to the node basis, described below;
- (iii) $I(C(\omega^{\omega^\alpha})) = \omega^{1+\alpha+1}$.

Notice that if $\alpha = n$ is finite, then $I_b(C(\omega^{\omega^n})) < I(C(\omega^{\omega^n}))$. Note also that since the unit vector basis for X_α is unconditional, it follows that $I_b^+(X_\alpha) = I_b(X_\alpha)$. Neither X_α , nor $C(\omega^{\omega^\alpha})$ is reflexive, so $I^+(X_\alpha) = I^+(C(\omega^{\omega^\alpha})) = \omega_1$.

In order to prove the above theorem we must first describe the node basis for $C(\omega^{\omega^\alpha})$ and clarify the relationship between the Schreier sets, the Schreier spaces X_α and $C(\omega^{\omega^\alpha})$.

We know that if we identify \mathcal{S}_α with $\{\mathbf{1}_F : F \in \mathcal{S}_\alpha\} \subset \{0, 1\}^\mathbb{N}$, then \mathcal{S}_α is homeomorphic to $[1, \omega^{\omega^\alpha}]$ in the topology of pointwise convergence (see [AA] and [MS]). Thus we shall consider this representation of $[1, \omega^{\omega^\alpha}]$ in the sequel.

Define a partial order on \mathcal{S}_α by $F \preceq G$ if and only if F is an initial segment of G , i.e. $G \cap \{1, 2, \dots, \max F\} = F$. This order induces a natural tree structure on \mathcal{S}_α . For each $F \in \mathcal{S}_\alpha$ define a function $\chi_F : \mathcal{S}_\alpha \rightarrow \{0, 1\}$ by

$$\chi_F(G) = \begin{cases} 1, & \text{if } F \preceq G; \\ 0, & \text{otherwise.} \end{cases}$$

This function is thus 1 on every G in \mathcal{S}_α that extends F . Let $\mathcal{B}(\omega^{\omega^\alpha}) = \{\chi_F : F \in \mathcal{S}_\alpha\}$, then we say that $(\chi_{F_i})_{i=0}^\infty$ is an admissible enumeration of $\mathcal{B}(\omega^{\omega^\alpha})$ if and only if $F_i \prec F_j$ implies $i < j$. Since $\emptyset \preceq F$ for every $F \in \mathcal{S}_\alpha$, it follows that $F_0 = \emptyset$ in any admissible enumeration of $\mathcal{B}(\omega^{\omega^\alpha})$. Notice that admissible enumerations preserve the tree structure in that we have an order preserving map from $(\mathcal{S}_\alpha, \prec)$ to \mathbb{N} . Furthermore, for each $F \in \mathcal{S}_\alpha$ we have $\chi_F \in C(\omega^{\omega^\alpha})$. Indeed, if $(G_i)_{i=1}^\infty$ is a sequence in \mathcal{S}_α converging to G , then for every $N \geq 1$ there exists $I \geq 1$ such that $G_i \cap \{1, \dots, N\} = G \cap \{1, \dots, N\}$ for every $i \geq I$, and in particular there exists $I_G \geq 1$ such that $G_i \cap \{1, \dots, \max G\} = G$ for every $i \geq I_G$. It is now clear that $F \preceq G$ if and only if $F \preceq G_i$ for every $i \geq I_G$ and hence $\chi_F \in C(\omega^{\omega^\alpha})$ as required.

Lemma 5.6. *If $(\chi_{F_i})_{i=0}^\infty$ is an admissible enumeration of $\mathcal{B}(\omega^{\omega^\alpha})$, then $(\chi_{F_i})_{i=0}^\infty$ is a monotone basis for $C(\omega^{\omega^\alpha})$.*

Proof. We first show that $(\chi_{F_i})_{i=0}^\infty$ is a monotone basic sequence, and then apply the Stone-Weierstrass theorem to obtain that its span is all of $C(\omega^{\omega^\alpha})$.

Let $(a_i)_{i=0}^\infty \in \mathbf{R}$, then

$$\left\| \sum_{i=0}^{k+1} a_i \chi_{F_i} \right\| = \sup_{G \in \mathcal{S}_\alpha} \left| \sum_{i=0}^{k+1} a_i \chi_{F_i}(G) \right| = \sup_{G \in \mathcal{S}_\alpha} \left| \sum_{\substack{i: F_i \preceq G \\ 0 \leq i \leq k+1}} a_i \right|.$$

Since \mathcal{S}_α is hereditary and the enumeration of $\mathcal{B}(\omega^{\omega^\alpha})$ is admissible, it follows that there is an index $i_0 \leq k$ and $j > F_{i_0}$ such that $F_{k+1} = F_{i_0} \cup \{j\}$. Next, observe that for all $G \in \mathcal{S}_\alpha$ with $F_{k+1} \preceq G$, if $G' = G \setminus \{j\}$, then for $i \leq k+1$, $F_i \preceq G'$ if and only if $F_i \preceq G$ and $i \leq k$ (since $G \cap F_{k+1} = G \cap \{1, \dots, j\} = F_{k+1}$). Thus

$$\sup_{G \in \mathcal{S}_\alpha} \left| \sum_{\substack{i: F_i \preceq G \\ 0 \leq i \leq k+1}} a_i \right| \geq \sup_{G \in \mathcal{S}_\alpha} \left| \sum_{\substack{i: F_i \preceq G \\ 0 \leq i \leq k}} a_i \right| = \left\| \sum_{i=0}^k a_i \chi_{F_i} \right\|,$$

and so $(\chi_{F_i})_{i=0}^\infty$ is a monotone basic sequence.

To see that $[\chi_F : F \in \mathcal{S}_\alpha] = C(\omega^{\omega^\alpha})$ we shall apply the Stone-Weierstrass theorem. Since $\chi_\emptyset = 1$ for each $G \in \mathcal{S}_\alpha$, it follows that $[\chi_F : F \in \mathcal{S}_\alpha]$ contains the constant function. It is easy to see that $[\chi_F : F \in \mathcal{S}_\alpha]$ separates the points of \mathcal{S}_α , so it remains to show that the set contains the algebra generated by $\{\chi_F : F \in \mathcal{S}_\alpha\}$. If $F, F' \in \mathcal{S}_\alpha$ and $\chi_F \cdot \chi_{F'}$ is not identically zero, then there exists $G \in \mathcal{S}_\alpha$ such that $\chi_F(G) = \chi_{F'}(G) = 1$, i.e., $F \preceq G$ and $F' \preceq G$ so that both $\{1, 2, \dots, \max F\} \cap G = F$, and $\{1, 2, \dots, \max F'\} \cap G = F'$. Hence either $F' \preceq F$, or $F \preceq F'$ which gives $\chi_F \cdot \chi_{F'} = \chi_F$ or $\chi_{F'}$ respectively. In either case we have that the algebra is contained in the linear span, as required, which completes the proof. \square

Definition 5.7. Since any admissible enumeration of $\mathcal{B}(\omega^{\omega^\alpha})$, is a monotone basis for $C(\omega^{\omega^\alpha})$, we shall call $\mathcal{B}(\omega^{\omega^\alpha})$ the *node basis* for $C(\omega^{\omega^\alpha})$.

Remark 5.8. For any point $\beta \in [1, \omega^{\omega^\alpha}]$ there are only finitely many elements in the node basis which have β in their support. With this in mind it is clear that the node basis is shrinking.

Finally let us consider the spaces X_α . We have defined

$$\left\| \sum a_i e_i \right\|_\alpha = \sup_{G \in \mathcal{S}_\alpha} \left| \sum_{i \in G} a_i \right|.$$

For each $i \geq 1$ let $f_i = \mathbf{1}_{\{G \in \mathcal{S}_\alpha : i \in G\}}$. Clearly $f_i \in C(\omega^{\omega^\alpha})$ for each i , and in $C(\omega^{\omega^\alpha})$:

$$\left\| \sum a_i f_i \right\|_{C(\omega^{\omega^\alpha})} = \sup_{F \in \mathcal{S}_\alpha} \left| \sum a_i f_i(F) \right| = \sup_{F \in \mathcal{S}_\alpha} \left| \sum_{i \in F} a_i \right| = \left\| \sum a_i e_i \right\|_\alpha.$$

Thus $(f_i)_1^\infty$ is 1-equivalent to $(e_i)_1^\infty$ and X_α can be isometrically embedded in $C(\omega^{\omega^\alpha})$. Actually, more is true.

Lemma 5.9. *If $(\chi_{F_i})_0^\infty$ is an admissible enumeration of $\mathcal{B}(\omega^{\omega^\alpha})$, then the basis $(f_i)_1^\infty$ for X_α is 1-equivalent to a block basis of $(\chi_{F_i})_0^\infty$.*

Proof. The heart of the proof lies in choosing an appropriate block basis of $(\chi_{F_i})_0^\infty$. To do this we shall construct a tree isomorphism ψ from \mathcal{S}_α into \mathcal{S}_α by induction, and then the map from the basis $(f_i)_1^\infty$ of X_α to a block basis of $(\chi_{F_i})_0^\infty$ will be given by

$$U(f_i) = \sum_{\substack{G \in \mathcal{S}_\alpha \\ \max G = i}} \chi_{\psi G}.$$

This immediately gives the ordering requirement on ψ that if $\max G = n$ and $\max G' = n+1$, then $\chi_{\psi G}$ precedes $\chi_{\psi G'}$, i.e. if $\psi G = F_i$ and $\psi G' = F_j$, then $i < j$.

To help us write down the construction of ψ more explicitly we define subtrees T_F of \mathcal{S}_α , for $F \in \mathcal{S}_\alpha$, by $T_F = \text{supp } \chi_F = \{G \in \mathcal{S}_\alpha : F \preceq G\}$, with the order \preceq , and as usual $o(T_F)$ is the order of the tree. Clearly $T_\emptyset = \mathcal{S}_\alpha$ and $o(T_F)$ is a successor ordinal for each $F \in \mathcal{S}_\alpha$, since F is the unique initial node.

Let F be a non-terminal node of the tree T_\emptyset , so that there exists $G \in \mathcal{S}_\alpha$ with $F \prec G$. Then there are infinitely many sets $G \in \mathcal{S}_\alpha$ with $F \prec G$ and $|G| = |F| + 1$. Thus, if $o(T_F) = \beta + 1$, then $o(T_G) \leq \beta$ for every such set G .

To simplify the notation for the induction we shall use an enumeration $(G_j)_0^\infty$ of \mathcal{S}_α , the domain of ψ , which satisfies: $G_0 = \emptyset$, $G_1 = \{1\}$, and if $\max G_j < \max G_k$, then $j < k$.

Now let us inductively define ψ . Let $\psi \emptyset = \emptyset = F_0$, $\psi G_1 = \psi \{1\} = F_1$, and set $k_1 = 1$. Suppose that ψG_i has been defined for $i \leq n$ such that if $i < j$, $\psi G_i = F_{k_i}$ and $\psi G_j = F_{k_j}$, then $k_i < k_j$, $G_i \prec G_j$ if and only if $F_{k_i} \prec F_{k_j}$ and for $i = 1, \dots, n$, $o(T_{G_i}) \leq o(T_{F_{k_i}})$. We next define ψG_{n+1} . From our enumeration $(G_j)_0^\infty$ of \mathcal{S}_α there exists $m \geq 1$ such that $G_{n+1} = \{m\}$, or $G_{n+1} = G_i \cup \{m\}$ for some $i \leq n$. In the first case let $k_{n+1} > k_n$ be the least integer such that $|F_{k_{n+1}}| = 1$ and $o(T_{F_{k_{n+1}}}) \geq o(T_{G_{n+1}})$. We can achieve this last condition because $\sup_{r \geq 1} o(T_{\{r\}}) = \omega^\alpha$. In the second case let $k_{n+1} > k_n$ be least with $F_{k_i} \prec F_{k_{n+1}}$, $|F_{k_{n+1}}| = |F_{k_i}| + 1$, and $o(T_{F_{k_{n+1}}}) \geq o(T_{G_{n+1}})$. The existence of k_{n+1} is guaranteed by the conditions on ψ . Indeed, since $G_{n+1} = G_i \cup \{m\}$, it follows that G_i is not a terminal node of T_\emptyset so that $o(T_{G_i}) > 1$. We also have $o(T_{F_{k_i}}) \geq o(T_{G_i})$, so

that neither is F_{k_i} a terminal node of T_\emptyset and hence F_{k_i} has an infinite sequence of successor nodes $(E_j)_1^\infty$ such that $E_j \in \mathcal{S}_\alpha$, $F_{k_i} \prec E_j$, $|E_j| = |F_{k_i}| + 1$ for each $j \geq 1$ and $\sup_j o(T_{E_j}) = o(T_{F_{k_i}}) - 1 = \beta$, where $o(T_{F_{k_i}}) = \beta + 1$. If β is a successor ordinal, then choose E_j so that

$$o(T_{E_j}) = \beta = o(T_{F_{k_i}}) - 1 \geq o(T_{G_i}) - 1 \geq o(T_{G_{n+1}}) .$$

Otherwise $o(T_{G_{n+1}}) < \beta = o(T_{G_i}) - 1$, and we may choose E_j so that $o(T_{E_j}) \geq o(T_{G_{n+1}})$. We then set $F_{k_{n+1}} = E_j$.

Clearly $\psi G_n = F_{k_n}$ ($n \geq 1$) satisfies all the requirements of the induction. It remains to show that this is sufficient to ensure that the map U is an isometry. We must show that $\|\sum a_i U(f_i)\|_{C(\omega^{\omega^\alpha})} = \sup_{F \in \mathcal{S}_\alpha} |\sum a_i|$. Now,

$$\begin{aligned} \left\| \sum a_i U(f_i) \right\|_{C(\omega^{\omega^\alpha})} &= \left\| \sum a_i \sum_{\substack{G_j \in \mathcal{S}_\alpha \\ \max G_j = i}} \chi_{\psi G_j} \right\|_{C(\omega^{\omega^\alpha})} \\ &= \sup_{F \in \mathcal{S}_\alpha} \left| \sum a_i \sum_{\substack{G_j \in \mathcal{S}_\alpha \\ \max G_j = i}} \chi_{\psi G_j}(F) \right|. \end{aligned}$$

First note that for i fixed, if there exist j, j' such that $\chi_{\psi G_j}(F) = \chi_{\psi G_{j'}}(F) = 1$, and $\max G_j = \max G_{j'} = i$, then both $\psi G_j \preceq F$ and $\psi G_{j'} \preceq F$ so that we may assume $\psi G_j \preceq \psi G_{j'}$. By the conditions on ψ this forces $G_j \preceq G_{j'}$, but $\max G_j = \max G_{j'}$ so that $G_j = G_{j'}$, and hence $j = j'$. Thus, for each $i \geq 1$, $F \in \mathcal{S}_\alpha$ we have $\sum_{G_j \in \mathcal{S}_\alpha, \max G_j = i} \chi_{\psi G_j}(F) = 0$ or 1 .

To complete the proof we define a map φ from \mathcal{S}_α into the collection of finite subsets on \mathbb{N} by

$$\varphi(F) = \left\{ i \geq 1 : \sum_{\substack{G_j \in \mathcal{S}_\alpha \\ \max G_j = i}} \chi_{\psi G_j}(F) = 1 \right\}$$

and show that the range of φ is \mathcal{S}_α , for then

$$\sup_{F \in \mathcal{S}_\alpha} \left| \sum a_i \sum_{\substack{G_j \in \mathcal{S}_\alpha \\ \max G_j = i}} \chi_{\psi G_j}(F) \right| = \sup_{F \in \mathcal{S}_\alpha} \left| \sum a_i \right|$$

as required.

First note that if $E \preceq F$, then $\varphi(E) \preceq \varphi(F)$. Indeed,

$$\begin{aligned} \varphi(F) &= \left\{ i \geq 1 : \sum_{\substack{G_j \in \mathcal{S}_\alpha \\ \max G_j = i}} \chi_{\psi G_j}(F) = 1 \right\} \\ &= \{ i \geq 1 : \text{there exists } j \geq 1 \text{ with } i = \max G_j \text{ and } \psi G_j \preceq F \} \\ &= \{ \max G_j : \psi G_j \preceq F \} . \end{aligned} \tag{*}$$

Now fix $j \geq 1$ and let $E_1 \prec \cdots \prec E_k = G_j$ satisfy $|E_1| = 1$ and $|E_{i+1}| = |E_i| + 1$ ($i < k$). Then $\psi E_i \preccurlyeq \psi G_j$, so that $\max E_i \in \varphi(\psi G_j)$ ($i = 1, \dots, k$), i.e. $G_j \subseteq \varphi(\psi G_j)$. On the other hand, if $\psi G_{j'} \preccurlyeq \psi G_j$, then $G_{j'} \preccurlyeq G_j$ which gives $G_{j'} = E_i$ for some i , so that $\varphi(\psi G_j) \subseteq G_j$ by (*), and hence the two sets are equal. Thus \mathcal{S}_α is contained in the range of φ . Finally let $F \in \mathcal{S}_\alpha$, set $i = \max \varphi(F)$ and find j_0 such that $i = \max G_{j_0}$ and $\psi G_{j_0} \preccurlyeq F$. But then $G_{j_0} = \varphi(\psi G_{j_0}) \preccurlyeq \varphi(F)$, while $\max G_{j_0} = i = \max \varphi(F)$. Hence $G_{j_0} = \varphi(F)$ and so the range of φ is exactly \mathcal{S}_α as required. This completes the proof. \square

Lemma 5.10. *For $\alpha < \omega_1$, $\omega^{\alpha+1} \leq I_b(X_\alpha) \leq I_b(C(\omega^{\omega^\alpha})) \leq \omega^{\alpha+2}$. Moreover, when $\omega \leq \alpha < \omega_1$, $I_b(X_\alpha) = I_b(C(\omega^{\omega^\alpha})) = \omega^{\alpha+1}$.*

Proof. First let $(e_i)_1^\infty$ be the unit vector basis for X_α and set $T = \{(e_i)_{i \in F} : F \in \mathcal{S}_\alpha\}$. The tree T is clearly an ℓ_1 -block basis tree on X_α isomorphic to $\text{Tree}(\mathcal{S}_\alpha) \setminus \{\emptyset\}$, so that $o(T) = \omega^\alpha$. By Lemma 3.5 the block basis index is strictly greater than the order of any block basis tree on the space, so that $\omega^\alpha < I_b(X_\alpha)$. But now, by Corollary 3.8, the block basis index is of the form ω^β for some $\beta < \omega_1$ so that $\omega^{\alpha+1} \leq I_b(X_\alpha)$.

As we noted after Theorem 5.4, $I_w^+(C(\omega^{\omega^\alpha})) = \eta(C(\omega^{\omega^\alpha})) = \omega^{\alpha+1}$, and since the node basis for $C(\omega^{\omega^\alpha})$ is shrinking, it follows from Theorem 3.23 that $I_b^+(C(\omega^{\omega^\alpha})) = \omega^{\alpha+1}$ when $\omega \leq \alpha$ and $I_b^+(C(\omega^{\omega^\alpha})) \leq \omega^{\alpha+2}$ when $\alpha = n < \omega$. It is clear that $I_b(C(\omega^{\omega^\alpha})) \leq I_b^+(C(\omega^{\omega^\alpha}))$, and finally we showed in Lemma 5.9 that X_α embeds into $C(\omega^{\omega^\alpha})$ as a block basis, and hence we have the inequalities

$$\begin{aligned} \omega^{\alpha+1} &\leq I_b(X_\alpha) \leq I_b(C(\omega^{\omega^\alpha})) \leq \omega^{\alpha+1} \text{ when } \omega \leq \alpha, \text{ and} \\ \omega^{\alpha+1} &\leq I_b(X_\alpha) \leq I_b(C(\omega^{\omega^\alpha})) \leq \omega^{\alpha+2} \text{ when } \alpha = n < \omega, \end{aligned}$$

which completes the proof. \square

Remark 5.11. Since $\omega \leq \alpha$, it follows from Theorem 3.14 (ii) that also $I(X_\alpha) = I(C(\omega^{\omega^\alpha})) = \omega^{\alpha+1}$.

Lemma 5.12. *For each $n \geq 1$, every $k \geq 1$ and any admissible enumerations of the node bases of $(C(\omega^{\omega^n}) \oplus \cdots \oplus C(\omega^{\omega^n \cdot k}))_\infty$ and $C(\omega^{\omega^n})$, the node basis of $(C(\omega^{\omega^n}) \oplus \cdots \oplus C(\omega^{\omega^n \cdot k}))_\infty$ embeds isomorphically into $C(\omega^{\omega^n})$ as a block basis of the node basis.*

Before we can prove this lemma we need to extend the definition of node basis from $C(\omega^{\omega^\alpha})$ to $C(\omega^{\omega^{\alpha \cdot k}})$ and $(C(\omega^{\omega^\alpha}) \oplus \cdots \oplus C(\omega^{\omega^{\alpha \cdot k}}))_\infty$ for $\alpha < \omega$, $k \geq 1$. First observe that in $\mathcal{S}_{\alpha+1}$ we have a

natural copy of $\omega^{\omega^\alpha \cdot k}$ given by

$$\mathcal{S}_{\alpha,k} = \{\{k+1\} \cup \bigcup_{i=1}^k F_i : \{k+1\} < F_1 < \dots < F_k, F_i \in \mathcal{S}_\alpha\}$$

so that $\{\mathbf{1}_{\{G:F \preceq G\}} = \chi_F : F \in \mathcal{S}_{\alpha,k}\}$ is the node basis for $C(\omega^{\omega^\alpha \cdot k})$. Further, $\mathcal{S}_{\alpha,k} \cap \mathcal{S}_{\alpha,l} = \emptyset$ if $k \neq l$, thus $\{\chi_{\{l\}}\}_{l=2}^{k+1}$ is a sequence of disjointly supported functionals and $\{\chi_F : \{l\} \preceq F, 2 \leq l \leq k+1, F \in \mathcal{S}_{\alpha+1}\}$ is a node basis for

$$(C(\omega^{\omega^\alpha}) \oplus \dots \oplus C(\omega^{\omega^\alpha \cdot k}))_\infty = \{f \in C(\omega^{\omega^{\alpha+1}}) : f(F) = 0 \text{ if there exists } j > k+1 \text{ with } \{j\} \preceq F\}.$$

The natural projection Q_k of $C(\omega^{\omega^{\alpha+1}})$ onto $(C(\omega^{\omega^\alpha}) \oplus \dots \oplus C(\omega^{\omega^\alpha \cdot k}))_\infty$ is given by

$$Q_k g = \left(\sum_{l=2}^{k+1} \chi_{\{l\}} \right) \cdot g.$$

Finally we note that if $(e_i)_0^\infty$ is an admissible ordering of the node basis of $C(\omega^{\omega^\alpha})$, then $e_0 = \chi_\emptyset = \mathbf{1}_{[1, \omega^{\omega^\alpha}]}$, and hence $(e_i)_1^\infty$ is a node basis for $C_0(\omega^{\omega^\alpha}) = \{f \in C(\omega^{\omega^\alpha}) : f(\omega^{\omega^\alpha}) = 0\}$.

Proof of Lemma 5.12. The argument follows the same lines as the proof that $C(\omega^{\omega^\alpha \cdot k})$ is isomorphic to $C(\omega^{\omega^\alpha})$ in [BP]. Note that for $\alpha = 0$ the node basis of $(C(\omega) \oplus \dots \oplus C(\omega^k))_\infty$ is a family of indicator functions with nested or disjoint supports, and the nested functions are at most $k+1$ sets deep. The required map T is found by sending the i^{th} element of the admissible enumeration of the node basis of $(C(\omega) \oplus \dots \oplus C(\omega^k))_\infty$ to the $(i+1)^{\text{th}}$ element $\chi_{\{i\}}$ of the node basis of $C(\omega)$, $(\chi_\emptyset, \chi_{\{1\}}, \chi_{\{2\}}, \dots)$. (Note that χ_\emptyset is not in the image.) It is easy to see that $\|T\| \cdot \|T^{-1}\| \leq 2(k+1)$. The general case is similar.

We view the node basis of $(C(\omega^{\omega^n}) \oplus \dots \oplus C(\omega^{\omega^n \cdot k}))_\infty$ (in the ordering \prec) as a disjoint union of k trees $\{\chi_{\{m+1\}}\} \cup T(m)$, $m = 1, 2, \dots, k$, with $T(m)$ isomorphic to the replacement tree $T(m, \omega^{\omega^n})$ and $\chi_{\{m+1\}}$ the unique initial node of the tree $\{\chi_{\{m+1\}}\} \cup T(m)$. Thus $z \in T(m)$ implies $z = \chi_F$ with $\{m+1\} \prec F$.

For each $m = 1, \dots, k$ let $F_m : T(m) \rightarrow T_m = \{a_1^m, \dots, a_m^m\}$ be the defining map for the replacement tree. Recall that $F_m^{-1}(a_i^m)$ is one or a countable union of trees, each isomorphic to $T_{\omega^{\omega^n}}$. Let $(U_j)_{j=1}^\infty$ be an enumeration of all of these trees for $1 \leq i \leq m \leq k$. For each j let $(\chi_{G_{jl}})_{l=1}^\infty$ be the sequence of initial nodes of U_j , so that $\{\chi_F \in U_j : G_{jl} \preceq F\}$ is equivalent to the node basis of $C(\omega^{\omega^{n-1} \cdot l})$ under the natural map. Let $(y_i)_1^\infty$ be the given admissible enumeration of the node basis of $(C(\omega^{\omega^n}) \oplus \dots \oplus C(\omega^{\omega^n \cdot k}))_\infty$ and let $(w_j)_1^\infty$ be an admissible enumeration of the node basis of $C(\omega^{\omega^n})$. To avoid confusion between domain and range we shall let $\zeta_F = \mathbf{1}_{\{G \in \mathcal{S}_n : F \preceq G\}}$, for $F \in \mathcal{S}_n$ denote the elements of the node basis of $C(\omega^{\omega^n})$ in the image. Thus $\{w_j : j \geq 1\} = \{\zeta_F : F \in \mathcal{S}_n\}$.

We define a map $\psi : (y_i)_1^\infty \rightarrow (w_j)_1^\infty$ inductively to satisfy the following conditions:

- (i) $w_1 = \zeta_\emptyset$ is not in the image of ψ ;
- (ii) if $y_i = \chi_E$ and $E = \{m\}$ ($m = 1, \dots, k$) or $E = G_{jl}$ for some $j, l \geq 1$, then $\psi(y_i) = \zeta_{\{s\}}$ for some $s \geq 1$;
- (iii) ψ is increasing, i.e. if $\psi(y_i) = w_{l(i)}$, then $l(1) < l(2) < \dots$;
- (iv) if $G_{jl} \prec E_1 \prec E_2$ and $\psi(\chi_{G_{jl}}) = \zeta_{F_0}$, $\psi(E_i) = \zeta_{F_i}$ ($i = 1, 2$), then $F_0 \prec F_1 \prec F_2$;
- (v) if $\psi(E) = \zeta_F$, $G_{jl} \prec E$ and $\{s\} \prec F$, then the order of χ_E in $\{\chi_H : G_{jl} \prec H, H \in U_j\}$ is less than or equal to the order of ζ_F in $\{\zeta_H : \{s\} \prec H\}$, where the sets are trees in the usual order \prec and the order of a node z in a tree T is simply the order of the subtree $\{y \in T : y \leq z\}$ of T .

It is easy to see that the inductive definition of ψ will succeed because if $\psi(y_1), \dots, \psi(y_i)$ have been chosen, then there are infinitely many candidates for $\psi(y_{i+1})$ satisfying (i)–(v). It is also not difficult to see that if S is the induced map from $(C(\omega^{\omega^n}) \oplus \dots \oplus C(\omega^{\omega^n \cdot k}))_\infty$ into $C(\omega^{\omega^n})$, then $\|S\| \cdot \|S^{-1}\| \leq 2(k+1)$. \square

Remark 5.13. It is clear from the proof that the blocking of the basis of $C(\omega^{\omega^n})$ is actually just a subsequence. The same argument works for $(C(\omega^{\omega^\alpha}) \oplus \dots \oplus C(\omega^{\omega^\alpha \cdot k}))_\infty$ into $C(\omega^{\omega^\alpha})$, and the argument also shows that the node basis of $C(\omega^{\omega^n})$ is equivalent to a subsequence of the node basis of $C_0(\omega^{\omega^n})$.

Lemma 5.14. *For $n \geq 1$, $I_b(X_n) = I_b(C(\omega^{\omega^n})) = \omega^{n+1}$.*

Proof. By Lemma 5.9 and the proof of Lemma 5.10 we have

$$\omega^{n+1} \leq I_b(X_n) \leq I_b(C(\omega^{\omega^n})) .$$

To complete the proof we show that for each $n \geq 0$ there does not exist an ℓ_1 -block basis tree on $C_0(\omega^{\omega^n})$ of order ω^{n+1} , and hence $I_b(C_0(\omega^{\omega^n})) \leq \omega^{n+1}$. Then, since $I_b(X, (e_i)_0^\infty) = I_b(X, (e_i)_1^\infty)$ for any space X with basis $(e_i)_0^\infty$, and $C_0(\omega^{\omega^n}) = [e_i]_1^\infty$ where $(e_i)_0^\infty$ is any admissible enumeration of the node basis for $C(\omega^{\omega^n})$, it follows that $I_b(C(\omega^{\omega^n})) = I_b(C_0(\omega^{\omega^n})) \leq \omega^{n+1}$.

We prove this result by induction on n . For $n = 0$ we first note that $C_0(\omega) = c_0$. Since the unit vector basis of c_0 does not contain ℓ_1^n 's uniformly as block bases, it follows that c_0 contains no ℓ_1 -block basis tree of order ω .

We assume that the result is true for n , and let $\{e_i : i \geq 1\}$ be an admissible enumeration of the node basis of $C_0(\omega^{\omega^{n+1}})$. Suppose that T is an ℓ_1 - K -block basis tree of order ω^{n+2} on $C_0(\omega^{\omega^{n+1}})$

which, without loss of generality, we assume consists of finitely supported vectors with respect to $(e_i)_1^\infty$, and is isomorphic to the minimal replacement tree $T(\omega, \omega^{n+1})$.

We write $T = \cup_{m=1}^\infty T(m)$, where $T(m)$ is a tree isomorphic to $T(m, \omega^{n+1})$ and the elements from different trees $T(m)$ are unrelated. Choose $m > 2K$ and let $F : T(m) \rightarrow T_m = \{a_1, \dots, a_m\}$, where $a_1 < a_2 < \dots < a_m$, be the defining map for the replacement tree $T(m, \omega^{n+1})$. Let $S_1 = F^{-1}(a_1)$ so that $S_1 \simeq T_{\omega^{n+1}}$. Let $(x_i^1)_{i=1}^{p_1}$ be a terminal node in S_1 . Define $x_1 = x_1^1$ and let

$$k_1 = \max\{k \geq 1 : Q_k x_1 \neq 0\}.$$

Let $S'_1 = \{z \in T(m) : (x_i^1)_{i=1}^{p_1} < z\}$, so that $S'_1 \cap F^{-1}(a_2)$ is isomorphic to $T_{\omega^{n+1}}$, and let S_2 be the restricted tree $R(S'_1 \cap F^{-1}(a_2))$. The tree S_2 is an ℓ_1 -block basis tree of order ω^{n+1} . By Lemma 5.12 and the induction hypothesis there is no ℓ_1 -block basis tree on $Q_{k_1}(C_0(\omega^{\omega^{n+1}}))$ of order ω^{n+1} . Consider the tree

$$Q_{k_1} S_2 = \{(Q_{k_1} z_1, \dots, Q_{k_1} z_l) : (z_1, \dots, z_l) \in S_2\};$$

note that $\|Q_{k_1} z_i\| \leq \|Q_{k_1}\| \|z_i\| \leq 1$ and $\text{supp } Q_{k_1} z_i \cap Q_{k_1} z_j = \emptyset$ when $i \neq j$ and $(z_i)_1^l \in S_2$ (since S_2 is a block basis tree). If there exists $\delta > 0$ such that for every $(z_i)_1^l \in S_2$ and $(b_i)_1^l \subset \mathbf{R}$,

$$\left\| \sum_{i=1}^l a_i Q_{k_1} z_i \right\| \geq \delta \sum_{i=1}^l |b_i|,$$

then the tree

$$\{(Q_{k_1} z_1 / \|Q_{k_1} z_1\|, \dots, Q_{k_1} z_l / \|Q_{k_1} z_l\|) : (z_i)_1^l \in S_2\}$$

would be an ℓ_1 - δ^{-1} -block basis tree on $Q_{k_1} C_0(\omega^{\omega^{n+1}})$ of order ω^{n+1} , contradicting the induction hypothesis. Therefore there is a terminal node $(x_i^2)_{i=1}^{p_2} \in S_2$ and $(b_i^2)_{i=1}^{p_2} \subset \mathbf{R}$ such that $\sum_{i=1}^{p_2} |b_i^2| = 1$ and $\|\sum_{i=1}^{p_2} b_i^2 Q_{k_1} x_i^2\| < 1/m$. Define $x_2 = \sum_{i=1}^{p_2} b_i^2 x_i^2$ and let $k_2 = \max\{k \geq 1 : Q_k x_2 \neq 0\} \vee (k_1 + 1)$.

As before we consider the tree $S'_2 = \{z \in T(m) : (x_1^1, \dots, x_{p_1}^1, x_1^2, \dots, x_{p_2}^2) < z\}$, so that $S'_2 \cap F^{-1}(a_3)$ is isomorphic to $T_{\omega^{n+1}}$, and we let S_3 be the restricted tree $R(S'_2 \cap F^{-1}(a_3))$. Arguing as above there is a terminal node $(x_i^3)_{i=1}^{p_3} \in S_3$ and $(b_i^3)_{i=1}^{p_3} \subset \mathbf{R}$ such that $\sum_{i=1}^{p_3} |b_i^3| = 1$ and setting $x_3 = \sum_{i=1}^{p_3} b_i^3 x_i^3$ gives $\|Q_{k_2} x_3\| < 1/m$. Continuing in this way we get $(x_i)_1^m$ a block basis of some node $y = (x_1^1, \dots, x_{p_1}^1, x_1^2, \dots, x_{p_2}^2, \dots, x_1^m, \dots, x_{p_m}^m)$ of $T(m)$ such that $x_i = \sum_{j=1}^{p_i} b_j^i x_j^i$ for some sequence $(b_j^i)_{j=1}^{p_i} \subset \mathbf{R}$ with $\sum_{j=1}^{p_i} |b_j^i| = 1$, together with a sequence $(k_i)_1^m$ such that $k_i = \max\{k \geq 1 : Q_k x_i \neq 0\} \vee (k_{i-1} + 1)$ (where $k_0 = 0$).

Let $x = \frac{1}{m} \sum_{i=1}^m x_i$, so that

$$\|x\| = \left\| \frac{1}{m} \sum_{i=1}^m x_i \right\| = \left\| \frac{1}{m} \sum_{i=1}^m \sum_{j=1}^{p_i} b_j^i x_j^i \right\| \geq \frac{1}{K} \frac{1}{m} \sum_{i=1}^m \sum_{j=1}^{p_i} |b_j^i| \geq \frac{1}{K} \frac{1}{m} \sum_{i=1}^m 1 = \frac{1}{K},$$

since T was an ℓ_1 - K -tree. On the other hand (with $Q_{k_0} = 0$)

$$\begin{aligned} \|x\| &= \max_{1 \leq j \leq m} \|(Q_{k_j} - Q_{k_{j-1}})x\| = \max_{1 \leq j \leq m} \left\| (Q_{k_j} - Q_{k_{j-1}}) \frac{1}{m} \sum_{i=1}^m x_i \right\| \\ &\leq \max_{1 \leq j \leq m} \frac{1}{m} \left(\|x_j\| + \sum_{i=j+1}^m \|Q_{k_{i-1}} x_i\| \right) \leq \max_{1 \leq j \leq m} \frac{1}{m} \left(1 + \frac{m-j}{m} \right) < \frac{2}{m} < \frac{1}{K}. \end{aligned}$$

Thus there exists no such tree T of order ω^{n+2} on $C_0(\omega^{\omega^{n+1}})$ which completes the proof. \square

The goal of the next few results is to show that the ℓ_1 -index of $C(\omega^{\omega^n})$ is ω^{n+2} . First we need some preliminary results.

Lemma 5.15. *Let K be a compact Hausdorff space, let $\kappa : C(K) \hookrightarrow C(K \times \{1, \dots, 2^n\})$ be the map $(\kappa f)(k, j) = f(k)$, let $\iota : C(\{1, \dots, 2^n\}) \hookrightarrow C(K \times \{1, \dots, 2^n\})$ be the map $(\iota f)(k, j) = f(j)$, and let $(r_m)_1^n$ be the standard Rademacher functions on $\{1, \dots, 2^n\}$ so that $(r_m)_1^n$ is 1-equivalent to the unit vector basis of ℓ_1^n . Then for every $f \in C(K)$ and any sequence $(a_m)_1^n \subset \mathbf{R}$,*

$$\left\| \kappa f + \iota \sum_{m=1}^n a_m r_m \right\| = \|f\| + \sum_{m=1}^n |a_m|.$$

Proof. Find k_0 so that $|f(k_0)| = \|f\|$, let $\varepsilon = \text{sign}(f(k_0))$, and find $l_0 \in \{1, \dots, 2^n\}$ such that $\sum_{m=1}^n a_m r_m(l_0) = \varepsilon \sum_{m=1}^n |a_m|$. Now,

$$\begin{aligned} \left| (\kappa f)(k_0, l_0) + \left(\iota \sum_{m=1}^n a_m r_m \right)(k_0, l_0) \right| &= \left| f(k_0) + \sum_{m=1}^n a_m r_m(l_0) \right| \\ &= \left| \varepsilon \|f\| + \varepsilon \sum_{m=1}^n |a_m| \right| \\ &= \|f\| + \sum_{m=1}^n |a_m|, \end{aligned}$$

as required. \square

Note that this result will also apply if we replace $C(K)$ with $C_0(\omega^\alpha)$, since functions on $C_0(\omega^\alpha)$ attain their norm.

Lemma 5.16. *Let $1 \leq \gamma < \alpha < \omega_1$; if there exists an ℓ_1 -tree with constant 1 and order β on $C_0(\omega^\alpha)$, then there exists an ℓ_1 -tree on $C_0(\omega^{\alpha+\gamma})$ with constant 1 and order $\beta + \omega \cdot \gamma$.*

Proof. We may write $C_0(\omega^{\alpha+1})$ as

$$C_0(\omega^{\alpha+1}) = \left(\sum_{j=1}^{\infty} \oplus C_0(\omega^\alpha \times (2^j, 2^{j+1}]) \right)_{c_0} = \left(C_0(\omega^\alpha \times (2, 4]) \oplus C_0(\omega^\alpha \times (4, 8]) \oplus \dots \right)_{c_0},$$

where $(2^j, 2^{j+1}] = \{2^j + 1, 2^j + 2, \dots, 2^{j+1}\}$. We shall prove the result using induction on γ .

Let $\gamma = 1$ and let $\kappa_j : C_0(\omega^\alpha) \hookrightarrow C_0(\omega^\alpha \times (2^j, 2^{j+1}])$ be the map $(\kappa_j f)(\beta, l) = f(\beta)$, restricted from $C(\omega^\alpha)$ to $C_0(\omega^\alpha)$ ($j = 1, 2, \dots$), let $(\bar{r}_i^j)_{i=1}^j$ be the Rademacher functions on $(2^j, 2^{j+1}]$ and let $(r_i^j)_{i=1}^j$ be the extension of these to $C_0(\omega^\alpha \times (2^j, 2^{j+1}])$ with $r_i^j(\beta, l) = \bar{r}_i^j(l)$. Let T be a tree with constant 1 and order β on $C_0(\omega^\alpha)$; we construct a tree of order $\beta + \omega$ on $C_0(\omega^{\alpha+1})$. Let

$$S_j = \{(r_1^j), (r_1^j, r_2^j), \dots, (r_1^j, \dots, r_j^j)\} \cup \{(r_1^j, \dots, r_j^j, \kappa_j x_1, \dots, \kappa_j x_m) : (x_i)_1^m \in T\},$$

and let $S = \bigcup_{j=1}^\infty S_j$ with the usual ordering by extension. The subtree of S given by $\bigcup_{j=1}^\infty \{(r_1^j), (r_1^j, r_2^j), \dots, (r_i^j)_{i=1}^j\}$ has order ω , and after every terminal node is a tree of order β so that $o(S) = \beta + \omega$. It is clear from the previous lemma that S is an ℓ_1 -tree with constant 1.

If the result is true for γ , then given an ℓ_1 -1-tree on $C_0(\omega^\alpha)$ of order β , there exists an ℓ_1 -1-tree on $C_0(\omega^{\alpha+\gamma})$ of order $\beta + \omega \cdot \gamma$, but now by the case $\gamma = 1$ there exists an ℓ_1 -1-tree on $C_0(\omega^{\alpha+(\gamma+1)}) = C_0(\omega^{(\alpha+\gamma)+1})$ of order $(\beta + \omega \cdot \gamma) + \omega = \beta + \omega \cdot (\gamma + 1)$.

Finally, if γ is a limit ordinal and the result has been proven for every $\gamma' < \gamma$, then let $\gamma_n \nearrow \gamma$ and

$$C_0(\omega^{\alpha+\gamma}) \simeq \left(C_0(\omega^{\alpha+\gamma_1}) \oplus C_0(\omega^{\alpha+\gamma_2}) \oplus \dots \right)_{c_0},$$

hence we may take the union of ℓ_1 -1-trees S_n on $C_0(\omega^{\alpha+\gamma_n})$ of order $\beta + \omega \cdot \gamma_n$ to obtain a tree on $C_0(\omega^{\alpha+\gamma})$ of order $\sup_n (\beta + \omega \cdot \gamma_n) = \beta + \omega \cdot \gamma$ as required. \square

Lemma 5.17. $I(C(\omega^{\omega^n})) = \omega^{n+2}$.

Proof. From Theorem 3.14 (iii) and Lemma 5.14 we know that the ℓ_1 -index of $C(\omega^{\omega^n})$ is either ω^{n+1} or ω^{n+2} and hence $I(C_0(\omega^{\omega^n})) = \omega^{n+1}$ or ω^{n+2} . For each $n \geq 0$ we shall construct an ℓ_1 -tree on $C_0(\omega^{\omega^n})$ of order ω^{n+1} so that $I(C_0(\omega^{\omega^n})) = \omega^{n+2}$ by Lemma 3.5 and the result follows. This is clear for $n = 0$ since ℓ_1^n embeds isometrically into $\ell_\infty^{2^n}$ for each $n \geq 1$, which immediately yields an ℓ_1 -1-tree of order ω .

We may now complete the proof by induction on n . If there is an ℓ_1 -1-tree on $C_0(\omega^{\omega^n})$ of order ω^{n+1} , then by the previous lemma there exists a tree of order $\omega^{n+1} \cdot k = \omega^{n+1} + \omega \cdot (\omega^n \cdot (k-1))$ on $C_0(\omega^{\omega^n \cdot k}) = C_0(\omega^{\omega^n + \omega^n \cdot (k-1)})$ for every $k \geq 1$. Taking the union over k of these we obtain an ℓ_1 -1-tree on $C_0(\omega^{\omega^{n+1}})$ of order ω^{n+2} as required. This completes the inductive step and hence the proof. \square

Lemma 5.18. $I(X_n) = \omega^{n+1}$.

Proof. Again, from Theorem 3.14 (ix) and Lemma 5.14 we know that $I(X_n)$ is either ω^{n+1} or ω^{n+2} . To demonstrate that it is the former we show that for each $n \geq 1$ there does not exist an ℓ_1 -tree on X_n of order ω^{n+1} .

We prove this by induction on n based on the following lemmas. The idea of the proof is that if we do have an ℓ_1 -tree of order ω^{n+1} on X_n , then we can find a node in that tree which admits an absolute convex combination with arbitrarily small norm. This contradicts the hypothesis that it was an ℓ_1 -tree. \square

Below, if $x \in X_1 = [e_i]$, with $x = \sum a_i e_i$, then we define the supremum norm of x to be $\|x\|_\infty = \sup |a_i|$.

Lemma 5.19. *For each $\varepsilon > 0$ and each $K \geq 1$ there exists $n \geq 1$ such that if $(x_i)_1^n$ is a sequence of norm one vectors in X_1 which is K -equivalent to the unit vector basis of ℓ_1^n , then there exists a norm one vector $x \in S([x_i]_1^n)$ with $\|x\|_\infty < \varepsilon$.*

Proof. Fix n and let $(x_i)_1^n$ be as in the statement of the lemma. Suppose that $\|x\|_\infty \geq \varepsilon$ for each $x \in S([x_i]_1^n)$. Then $\|x\|_{X_1} \geq \|x\|_\infty \geq \varepsilon \|x\|_{X_1}$ for each $x \in [x_i]$. We may assume that each x_i has finite support with respect to the unit vector basis of X_1 , and let $N = \max\{\text{supp}(x_i) : i \leq n\}$. Thus $([x_i]_1^n, \|\cdot\|_{X_1})$ embeds into ℓ_∞^N with constant $1/\varepsilon$ via the map $\hat{\cdot} : x \mapsto (e_j^*(x))_{j=1}^N$, and hence $(\hat{x}_i)_1^n$ has a lower ℓ_1 estimate with constant ε/K .

By James [Ja1], for fixed k and $\delta > 0$, if n is sufficiently large, then there exists a normalized block basis $(\hat{y}_i)_1^k$ of $(\hat{x}_i)_1^n$ such that $(\hat{y}_i)_1^k \stackrel{1+\delta}{\sim} \text{uvb } \ell_1^k$. Now if we take δ to be very small, depending on k , then we see that for each $i = 1, \dots, k$ the size of one of the sets $E_i = \{j \leq N : \hat{y}_i(j) > 1/2\}$ and $F_i = \{j \leq N : \hat{y}_i(j) < -1/2\}$ must be at least 2^{k-2} . We calculate the norm of y_1 in X_1 supposing that $|E_1| \geq 2^{k-2}$. Let E be the second half of E_1 , so that if $E_1 = \{e_1, \dots, e_r\}$, then $E = \{e_s, e_{s+1}, \dots, e_r\}$, where $s = [(r+1)/2]$. Clearly $E \in \mathcal{S}_1$, $|E| \geq \frac{1}{2}2^{k-2} = 2^{k-3}$ and

$$\|y_1\|_{X_1} \geq \left| \sum_{j \in E} y_1(j) \right| = \left| \sum_{j \in E} \hat{y}_1(j) \right| \geq |E| \cdot \frac{1}{2} \geq \frac{1}{2}2^{k-3} = 2^{k-4}.$$

On the other hand $\|y_1\|_{X_1} \leq \frac{1}{\varepsilon} \|\hat{y}_1\|_\infty = \frac{1}{\varepsilon}$, so this is impossible for large k , and hence for n large enough.

This contradicts our initial assumption that $\|x\|_\infty \geq \varepsilon$ for each $x \in S([x_i]_1^n)$ and hence there exists $x \in S([x_i]_1^n)$ with $\|x\|_\infty < \varepsilon$. \square

Lemma 5.20. *If T is a tree on $B_{X_1} \setminus \{0\}$ of order ω , then for any $\varepsilon > 0$ there exist $(x_i)_1^n \in T$ and $(a_i)_1^n \subset \mathbf{R}$ with $\sum |a_i| = 1$ and $\|\sum_1^n a_i x_i\|_\infty < \varepsilon$.*

Proof. Choose $\varepsilon > 0$, set $K = 1/\varepsilon$, and let T be a tree on X_1 as above. If there exist $(x_i)_1^n \in T$ and $(a_i)_1^n \subset \mathbf{R}$ such that

$$\left\| \sum_1^n a_i x_i \right\| < \frac{1}{K} \sum_1^n |a_i| \cdot \|x_i\| ,$$

then set $a = \sum_1^n |a_i|$ and $\bar{a}_i = a_i/a$, and we have

$$\left\| \sum_1^n \bar{a}_i x_i \right\|_\infty \leq \left\| \sum_1^n \bar{a}_i x_i \right\| < \frac{1}{K} \sum_1^n \frac{|a_i|}{a} \cdot \|x_i\| \leq \frac{1}{K} \sum_1^n \frac{|a_i|}{a} = \frac{1}{K} = \varepsilon ,$$

while $\sum_1^n |\bar{a}_i| = 1$ as required.

Otherwise set $\bar{T} = \{(\bar{x}_i)_1^n : (x_i)_1^n \in T\}$, where $\bar{x} = x/\|x\|$. Then \bar{T} is an ℓ_1 - K -tree on X of order ω , and from the previous lemma can find $(\bar{x}_i)_1^n \in \bar{T}$ and $x = \sum_1^n a_i \bar{x}_i \in S([\bar{x}_i])$ such that $\|x\|_\infty < \varepsilon$. Now, $\|x\| = 1$ and $\|x\| \leq \sum |a_i|$ so that $\sum_1^n |a_i| \geq 1$; also $\|x_i\| \leq 1$ for each i , thus if we set $a = \sum_1^n (|a_i|/\|x_i\|)$, then $a \geq 1$. Clearly $\frac{1}{a} \sum_1^n a_i \bar{x}_i$ has supremum norm less than that of x so that $\|\frac{1}{a} \sum_1^n a_i \bar{x}_i\|_\infty < \varepsilon$. Finally

$$\frac{1}{a} \sum_1^n a_i \bar{x}_i = \sum_1^n \frac{a_i}{a \|x_i\|} x_i .$$

so that setting $b_i = a_i/(a \|x_i\|)$, we obtain $\frac{1}{a} \sum_1^n a_i \bar{x}_i = \sum b_i x_i$ with $\|\sum b_i x_i\| < \varepsilon$ and $\sum |b_i| = 1$ as required. \square

Lemma 5.21. *If T is a tree on $B_{X_n} \setminus \{0\}$ of order ω^{n+1} , and $\varepsilon > 0$, then there exist $(x_i)_1^n \in T$ and $(a_i)_1^n \subset \mathbf{R}$ with $\sum_1^n |a_i| = 1$ and $\|\sum_1^n a_i x_i\|_{X_n} < \varepsilon$.*

Proof. We shall prove the result by induction on n . Let $n = 1$ and let T be a tree on X_1 of order ω^2 satisfying the hypotheses of the lemma. We may assume that T consists of finitely supported vectors with respect to the basis of X_1 , and that T is isomorphic to the minimal tree $T(\omega, \omega)$. We write $T = \cup_m T(m)$ where $T(m)$ is a tree isomorphic to $T(m, \omega)$ and the elements from different trees $T(m)$ are unrelated.

Choose $m > 2/\varepsilon$, and let $F : T(m) \rightarrow T_m = \{a_1, \dots, a_m\}$, where $a_1 < \dots < a_m$, be the defining map for the replacement tree $T(m, \omega)$. Let $S_1 = F^{-1}(a_1)$, so that $S_1 \simeq T_\omega$. Let $(x_i^1)_1^{p_1}$ be any terminal node in S_1 . Define $x_1 = x_1^1$ and let $k_1 = \max(\text{supp } x_1)$.

Let $S'_1 = \{z \in T(m) : (x_i^1)_{i=1}^{p_1} < z\}$, so that $S'_1 \cap F^{-1}(a_2)$ is isomorphic to T_ω , and let S_2 be the restricted tree $R(S'_1 \cap F^{-1}(a_2))$. The tree S_2 on X_1 has order ω , and satisfies the conditions of the previous lemma, thus we may find a terminal node $(x_i^2)_{i=1}^{p_2} \in S_2$ and $(a_i^2)_{i=1}^{p_2} \subset \mathbf{R}$ such that $\sum_1^{p_2} |a_i^2| = 1$ and $\|\sum_1^{p_2} a_i^2 x_i^2\|_\infty < 1/(2k_1)$. Define $x_2 = \sum_1^{p_2} a_i^2 x_i^2$ and let $k_2 = \max(\text{supp } x_2) \vee (k_1 + 1)$.

As before we consider the tree $S'_2 = \{z \in T(m) : (x_1^1, \dots, x_{p_1}^1, x_1^2, \dots, x_{p_2}^2) < z\}$, so that $S'_2 \cap F^{-1}(a_3)$ is isomorphic to T_ω , and we let S_3 be the restricted tree $R(S'_2 \cap F^{-1}(a_3))$. Arguing as above there is a terminal node $(x_i^3)_{i=1}^{p_3} \in S_3$ and $(a_i^3)_{i=1}^{p_3} \subset \mathbf{R}$ such that $\sum_1^{p_3} |a_i^3| = 1$ and setting $x_3 = \sum_1^{p_3} a_i^3 x_i^3$ gives $\|x_3\| < 1/(4k_2)$.

Continuing in this way we get $(x_i)_1^m$ a block basis of some node $z = (x_1^1, \dots, x_{p_1}^1, \dots, x_1^m, \dots, x_{p_m}^m)$ of $T(m)$ such that $x_i = \sum_1^{p_i} a_j^i x_j^i$ for some sequence $(a_j^i)_{j=1}^{p_i} \subset \mathbf{R}$ with $\sum_1^{p_i} |a_j^i| = 1$, together with a sequence $(k_i)_1^m$ such that $k_i = \max(\text{supp } x_i) \vee (k_{i-1} + 1)$, where $k_0 = 0$, and $\|x_{i+1}\|_\infty < 1/(2^i k_i)$ ($i > 1$). Let $x = \frac{1}{m} \sum_{i=1}^m x_i$, so that for $E \in \mathcal{S}_1$, if $k = \min E \geq |E|$, and $i \leq m$ is chosen so that $k_{i-1} < k \leq k_i$, then

$$\begin{aligned} \left| \sum_{r \in E} e_r^*(x) \right| &\leq \frac{1}{m} \left(\|x_i\| + |E| \sum_{j=i+1}^m \|x_j\|_\infty \right) \\ &\leq \frac{1}{m} \left(1 + k_i \sum_{j=i+1}^m \frac{1}{2^{j-1} k_{j-1}} \right) \\ &\leq \frac{1}{m} \left(1 + k_i \sum_{j=i+1}^m \frac{1}{2^{j-1} k_i} \right) \\ &\leq \frac{2}{m}. \end{aligned}$$

Thus $\|x\|_{X_1} \leq 2/m < \varepsilon$ so that x is the vector we seek. This completes the proof in the case $n = 1$.

We next suppose the result has been proven for $n - 1$ and let T be a tree on X_n of order ω^{n+1} . We may assume consists of finitely supported vectors in X_n , is isomorphic to $T(\omega, \omega^n)$, and may be written as $\cup_m T(m)$ with $T(m) \simeq T(m, \omega^n)$. As before, choose $m > 2/\varepsilon$, and let $F : T(m) \rightarrow T_m = \{a_1, \dots, a_m\}$ be the defining map for the replacement tree $T(m, \omega^n)$. Define $S_1, (x_i^1)_{i=1}^{p_1}, x_1, k_1, S'_1$ as for the case $n = 1$. This time the tree $S_2 = R(S'_1 \cap F^{-1}(a_2))$ has order ω^n on X_n , but we may also consider it as a tree of order ω^n on $B_{X_{n-1}} \setminus \{0\}$, and hence it satisfies the conditions of the lemma for $n - 1$. Thus, by the induction hypothesis, there exists a terminal node $(x_i^2)_{i=1}^{p_2} \in S_2$ and $(a_i^2)_{i=1}^{p_2} \subset \mathbf{R}$ such that $\sum_1^{p_2} |a_i^2| = 1$ and $\|\sum_1^{p_2} a_i^2 x_i^2\|_{X_{n-1}} < 1/(2k_1)$.

Continuing in this way we obtain $(x_i)_1^m$ a block basis of a node $z = (x_1^1, \dots, x_{p_1}^1, \dots, x_1^m, \dots, x_{p_m}^m)$ of $T(m)$ such that $x_i = \sum_1^{p_i} a_j^i x_j^i$ for some sequence $(a_j^i)_{j=1}^{p_i} \subset \mathbf{R}$ with $\sum_1^{p_i} |a_j^i| = 1$, together with a sequence $(k_i)_0^m$ such that $k_0 = 0$, $k_i = \max(\text{supp } x_i) \vee (k_{i-1} + 1)$ and $\|x_{i+1}\|_{X_{n-1}} < 1/(2^i k_i)$.

Let $x = \frac{1}{m} \sum_{i=1}^m x_i$, let $E \in \mathcal{S}_n$, $k = \min E$ and choose $i \leq m$ so that $k_{i-1} < k \leq k_i$. We may write $E = \cup_{l=1}^k E_l$ where $E_l \in \mathcal{S}_{n-1}$ ($l = 1, \dots, k$) and $k \leq E_1 < \dots < E_k$. Now,

$$\begin{aligned} \left| \sum_{r \in E} e_r^*(x) \right| &\leq \frac{1}{m} \left(\|x_i\|_{X_n} + \left| \sum_{r \in E} e_r^* \sum_{j=i+1}^m x_j \right| \right) \leq \frac{1}{m} \left(1 + \sum_{l=1}^k \left| \sum_{r \in E_l} e_r^* \sum_{j=i+1}^m x_j \right| \right) \\ &\leq \frac{1}{m} \left(1 + \sum_{j=i+1}^m \sum_{l=1}^k \left| \sum_{r \in E_l} e_r^*(x_j) \right| \right) \leq \frac{1}{m} \left(1 + \sum_{j=i+1}^m k \|x_j\|_{X_{n-1}} \right) \\ &\leq \frac{1}{m} \left(1 + \sum_{j=i+1}^m \frac{k_i}{k_{j-1}} 2^{-j+1} \right) \leq \frac{1}{m} \left(1 + \sum_{j=i+1}^m 2^{-j+1} \right) \leq \frac{1}{m} (1 + 1) = \frac{2}{m}, \end{aligned}$$

and hence $\|x\|_{X_n} \leq 2/m < \varepsilon$ as required. This completes the proof. \square

Lemma 5.18 now follows and the proof of Theorem 5.5 is finished.

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DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OK 74078-1058, U.S.A.

E-mail address: `alspach@math.okstate.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI-COLUMBIA, COLUMBIA, MO 65211, U.S.A.

E-mail address: `rjudd@math.missouri.edu`

DEPARTMENT OF MATHEMATICS UNIVERSITY OF TEXAS AT AUSTIN, AUSTIN, TX 78712-1082, U.S.A.

E-mail address: `odell@math.utexas.edu`